PRICING OF PARISIAN OPTIONS FOR A JUMP-DIFFUSION MODEL WITH TWO-SIDED JUMPS

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Abstract. Using the solution of the one-sided exit problem, a procedure to price Parisian barrier options in a jump-diffusion model with two-sided exponential jumps is developed. By extending a method developed in Chesney, Jeanbanc and Yor (1997) for the diffusion case to this more general setup, we arrive at a numerical pricing algorithm that significantly outperforms Monte Carlo simulation for the prices of such products.

1. Introduction

A Parisian barrier option (or Parisian option for short) is an option that is knocked in/out if the underlying asset price process stays above/below a barrier for longer than a specified period $\gamma > 0$ during the lifetime of the option. Such option types can be a quite useful alternative to classical barrier options in risk management, as they are more robust to price manipulation of the underlying when the underlying is close to the barrier and, secondly, close to the barrier the hedging does not have such a large Gamma as in the case of classical barrier options. However, the determination of their price is quite challenging even for simple asset price models. For Parisian options of European type in a Black-Scholes model, Chesney et al. [10] derived a formula for the Laplace transform of the option price (with respect to maturity time T) using excursion theory, which then needs to be inverted numerically. In Schröder [24], this Laplace transform is further studied when at the starting time the underlying process is known to already be above the barrier for a given time. Lambart and Lelong [22] provide explicit expressions for the Laplace transform for several types of European Parisian options and study their numerical inversion to obtain the respective option prices. In Bernard et al. [5], a fast method for the numerical inversion of the Laplace transform is provided. An alternative to inverting the Laplace transform for obtaining the price is to use a partial differential equations (PDE) approach, see for instance Haber et al. [18] and Avellaneda and Wu [4] for a lattice method. In Chesney and Gauthier [9], American Parisian options are considered. Two sided Parisian barrier options are studied in [3], [12] and [23]. Further extensions with respect to barrier types are considered in [11], [15] and [16]. Cumulative Parisian options in a diffusion model were studied by Hugonnier [19], for cumulative exchange options see Chen and Suchanecki [8]. The first paper to study the pricing of Parisian options in a model with jumps seems to be Dassios and Wu [13], who consider a compound Poisson model with (only) downward exponential jumps, see also [14] for a potential alternative method in this setting.

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In this paper we consider the pricing problem of Parisian options in a jump-diffusion model with two-sided jumps and use a different approach to represent the Laplace transform of the price with respect to maturity time T as a combination of functions whose Laplace transforms with respect to duration time γ are explicitly available. Consequently, the option price can then be obtained by numerically inverting these Laplace transforms w.r.t. γ and T. As we will illustrate in the final section of the paper, the resulting numerical procedure to obtain the Parisian option price for this two-sided jumps model by far outperforms the alternative of simulating the price paths and applying Monte Carlo procedures. We will assume the jumps to be exponentially distributed. Then, by conditioning on the first time when the process exits a given level, one can exploit the solution of the one-sided exit problem given in Kou and Wang [20] to obtain efficient numerical schemes for the determination of the option price. Both on the methodological and on the practical side, our approach can be interpreted as an extension of the one for the diffusion setting given in Chesney et al. [10].

The paper is organized as follows. In Section 2 the underlying model is introduced and some auxiliary results are provided that will be useful later on. In Section 3 we derive a formula for the Laplace transform of the underlying process at maturity, which only consists of terms with known Laplace transform. In Section 4 we provide the representation for the Laplace transform of the Parisian option price, whose ingredients are then further investigated in Sections 5 and 6. Section 7 shows how our result reduces to the formulas of Chesney et al. [10] for the diffusion case, and Section 8 considers the particular case of a (pure) compound Poisson model with one-sided exponential jumps and compares the resulting formulas with the ones of Dassios and Wu [13] who use a different method for this model. Finally, in Section 9 we illustrate the effectiveness of our approach for numerical purposes.

2. Model definition and preliminaries

Consider a jump-diffusion process of the form

$$X_t = X_0 + \sigma W_t + \mu t + \sum_{i=1}^{N_t} Y_i,$$

where $\sigma > 0$, W_t is a Brownian motion with $W_0 = 0$, N_t is a homogeneous Poisson process with rate λ and (Y_i) are i.i.d. random variables with density function

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}}, \qquad p, q > 0,\ \ p+q = 1.$$

Here W_t, N_t and (Y_i) are independent. This model was popularized by Kou and Wang [20] and we will keep our notation close to theirs. Assume now that the price process of the underlying asset is given by $S_t = S_0 e^{X_t}$ under a risk-neutral measure. Since this asset price model is in general incomplete, one needs to specify the choice of risk-neutral measure and we assume here that risk-neutrality is achieved by an a priori mean shift, i.e. a suitably mean-shifted asset price model is calibrated to market data. Let $\eta_1 > 1$ to ensure that $\mathbb{E}[S_t] < \infty$.
In the sequel we will only consider the Parisian up-and-in call option with barrier at 1 for S_t (this corresponds to a barrier at 0 for X_t). Other types of options follow similarly or by symmetry. Let τ_γ be the first time when an excursion of X_t above 0 has lasted for at least γ units of time, i.e.

$$\tau_\gamma = \inf\{t | t - g_t > \gamma\} \quad \text{with} \quad g_t = \sup\{s \leq t | X_s \leq 0\},$$

where $\sup\{\emptyset\} = 0$. The price of such an option under a constant interest rate $r > 0$ is then given by

$$C(\gamma, k, S_0, T) := e^{-rT}\mathbb{E}[(S_T - k)^+; \tau_\gamma < T] = e^{-rT}\mathbb{E}^x[(e^{X_T} - k)^+; \tau_\gamma < T],$$

where $\mathbb{E}^x$ means that we are considering the process X_t under the condition $X_0 = x$. The crucial quantity for the pricing of this option is the Laplace transform (with respect to T) of

$$e^{-rT}\mathbb{P}^x[X_T \in dy; \tau_\gamma < T],$$

i.e. the distribution of X_T on the set $\tau_\gamma < T$ (the discount term e^{-rT} is then just a multiplicative factor). Chesney et al. [10] determined this Laplace transform explicitly for the Black-Scholes model. In this paper we will study this quantity for the above more general model with two-sided exponential jumps and express $\mathbb{P}^x[X_T \in dy; \tau_\gamma < T]$ as a combination of functions whose Laplace transforms with respect to the duration time γ are available. Similarly to Labart and Lelong [22], we will subsequently integrate these functions with respect to y and use two successive Laplace inversions to obtain the option price.

At first let us recall some results for the process X from Kou and Wang [20]. The process X has a generator

$$Lu(x) = \frac{1}{2}\sigma^2 u''(x) + \mu u'(x) + \lambda\int_{-\infty}^{\infty} dy\,[u(x+y) - u(x)]f_Y(y).$$

For any $\theta \in (-\eta_2, \eta_1)$,

$$\mathbb{E}[e^{\theta X_t}] = e^{G(\theta)t},$$

where

$$G(\theta) = \theta\mu + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1\right).$$

For $\alpha > 0$, the equation

$$G(x) = \alpha$$

has exactly four solutions $\beta_1, \beta_2, -\beta_3, -\beta_4$ such that

$$0 < \beta_1 < \eta_1 < \beta_2, \text{ and } 0 < \beta_3 < \eta_2 < \beta_4. \tag{1}$$

Note that solutions to $G(x) = \alpha$ can be worked out explicitly. Write $\beta_1^*, \beta_2^*, -\beta_3^*, -\beta_4^*$ for the corresponding solutions for the dual process $X_t^* := -X_t$. Note that $\beta_1 = \beta_3^*, \beta_2 = \beta_4^*, \beta_3 = \beta_1^*, \beta_4 = \beta_2^*$.

Define $\nu_b^+ := \inf\{t \geq 0 : X_t > b\}$ as the first passage time of the process X_t above the barrier b; and similarly $\nu_b^- := \inf\{t \geq 0 : X_t < b\}$ as the first passage time of the process X_t below the barrier b with the convention $\inf\emptyset = \infty$. For such a jump-diffusion with parameters $(\sigma, \mu, \eta_1, \eta_2)$ and $b > 0$, set

$$\Gamma_1(\alpha, b) := \mathbb{E}^0\left[e^{-\alpha\nu_b^+}; X_{\nu_b^+} = b\right] = \frac{\eta_1 - \beta_1}{\beta_2 - \beta_1}e^{-b\beta_1} - \frac{\eta_1 - \beta_2}{\beta_2 - \beta_1}e^{-b\beta_2},$$

$$\widehat{\Gamma}_1(\alpha, s) := \int_0^\infty du\; se^{-su}\Gamma_1(\alpha, u) = \frac{s}{\beta_2 - \beta_1}\left(\frac{\eta_1 - \beta_1}{s + \beta_1} - \frac{\eta_1 - \beta_2}{s + \beta_2}\right)$$

and

$$\Gamma_2(\alpha, b) := \mathbb{E}^0\left[e^{-\alpha\nu_b^+}; X_{\nu_b^+} > b\right] = \frac{(\eta_1-\beta_1)(\eta_1-\beta_2)}{\eta_1(\beta_2-\beta_1)}\left(e^{-b\beta_2} - e^{-b\beta_1}\right),$$

$$\widehat{\Gamma}_2(\alpha, s) := \int_0^\infty du\, se^{-su}\Gamma_2(\alpha, u) = \frac{(\eta_1-\beta_1)(\eta_1-\beta_2)}{\eta_1(\beta_2-\beta_1)}\left(\frac{s}{s+\beta_2} - \frac{s}{s+\beta_1}\right).$$

Then

$$\mathbb{E}^0\left[e^{-\alpha\nu_b^+}; X_{\nu_b^+} - b \in dy\right]\Big/ dy = \Gamma_2(\alpha, b)\eta_1 e^{-\eta_1 y},\ y > 0.$$

Put

$$\Lambda_1(\alpha) := \lim_{\varepsilon\to 0+} \frac{1-\Gamma_1(\alpha,\varepsilon)}{\varepsilon} = \beta_1 + \beta_2 - \eta_1$$

and

$$\Lambda_2(\alpha) := \lim_{\varepsilon\to 0+} \frac{\Gamma_2(\alpha,\varepsilon)}{\varepsilon} = -\frac{(\eta_1-\beta_1)(\eta_1-\beta_2)}{\eta_1}.$$

Similarly, define $\Gamma_1^*(\alpha, b), \widehat{\Gamma}_1^*(\alpha, s), \Gamma_2^*(\alpha, b), \widehat{\Gamma}_2^*(\alpha, b), \Lambda_1^*(\alpha), \Lambda_2^*(\alpha)$ for the process $X^* = -X$. We will need some estimates involving the stopping time ν_ε^+ and ν_ε^-.

Lemma 2.1.

$$\lim_{\varepsilon\to 0+} \mathbb{E}^0[\nu_{-\varepsilon}^- \wedge \nu_\varepsilon^+]/\varepsilon^2 = C > 0 \tag{2}$$

and

$$\lim_{\varepsilon\to 0+} \mathbb{P}^0\{X_{\nu_{-\varepsilon}^-\wedge\nu_\varepsilon^+} \le -\varepsilon\} = 1/2. \tag{3}$$

Proof. These results are clearly true when the process X is a linear Brownian motion, i.e. $\lambda = 0$. This follows from Formulas 3.01 and 3.05 on page 309 of [6]. Intuitively, they are also true for $\lambda > 0$ since for small ε, the probability is of the order $O(\varepsilon^2)$ for the first jump of X to arrive before X exits from interval $[-\varepsilon, \varepsilon]$ due to the linear Brownian motion. For a formal proof one needs the solution to the two-sided exit problem for X, which can be found in Theorem 3 of [21]. □

We need the following key estimates.

Lemma 2.2.

$$\mathbb{P}^0\{\tau_\gamma < \infty, \tau_\gamma \le \nu_\varepsilon^+\} = O(\varepsilon^2). \tag{4}$$

Proof. Let (θ_t) denote the shift operator for process X. Define the consecutive down-crossing and up-crossing times of level $-\epsilon$ and level 0 as follows. $T_0^+ := 0, T_0^- := \nu_{-\varepsilon}^-, T_1^+ := T_0^- + \nu_0^+ \circ \theta_{T_0^-}, \ldots, T_i^- := T_i^+ + \nu_{-\varepsilon}^- \circ \theta_{T_i^+}, T_{i+1}^+ := T_i^- + \nu_0^+ \circ \theta_{T_i^-}, \ldots$. Consider the total number of up-crossings before time τ_γ and whether there is a down-crossing between τ_γ and ν_ε^+. Then applying the strong Markov property at time T_i^+ we have

$$\begin{aligned}
\mathbb{P}^0\{\tau_\gamma < \infty, \tau_\gamma \le \nu_\varepsilon^+\} &\le \sum_{i=0}^{\infty} \mathbb{P}^0\left\{T_i^+ < \tau_\gamma \le T_i^- < \nu_\varepsilon^+\right\} + \sum_{i=0}^{\infty} \mathbb{P}^0\left\{T_i^+ < \tau_\gamma \le \nu_\varepsilon^+ < T_i^-\right\} \\
&\le \sum_{i=0}^{\infty} \mathbb{E}^0\left[\mathbb{P}^{X_{T_i^+}}\left\{\tau_\gamma \le \nu_{-\varepsilon}^- < \nu_\varepsilon^+\right\}; T_i^+ < \nu_\varepsilon^+\right] \\
&\quad + \sum_{i=0}^{\infty} \mathbb{E}^0\left[\mathbb{P}^{X_{T_i^+}}\left\{\tau_\gamma \le \nu_\varepsilon^+ < \nu_{-\varepsilon}^-\right\}; T_i^+ < \nu_\varepsilon^+\right] \\
&= \sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} = 0\}\mathbb{P}^0\left\{\tau_\gamma \le \nu_{-\varepsilon}^- < \nu_\varepsilon^+\right\} \\
&\quad + \sum_{i=0}^{\infty} \int_{0+}^{\varepsilon} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} \in dx\}\mathbb{P}^x\left\{\tau_\gamma \le \nu_{-\varepsilon}^- < \nu_\varepsilon^+\right\} \\
&\quad + \sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} = 0\}\mathbb{P}^0\left\{\tau_\gamma \le \nu_\varepsilon^+ < \nu_{-\varepsilon}^-\right\} \\
&\quad + \sum_{i=0}^{\infty} \int_{0+}^{\varepsilon} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} \in dx\}\mathbb{P}^x\left\{\tau_\gamma \le \nu_\varepsilon^+ < \nu_{-\varepsilon}^-\right\}.
\end{aligned}$$

It follows that for ε small enough

$$\begin{aligned}
\mathbb{P}_0\{\tau_\gamma < \infty, \tau_\gamma \le \nu_\varepsilon^+\} &\le \sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} = 0\}\mathbb{P}^0\left\{\gamma \le \nu_\varepsilon^+ \wedge \nu_{-\varepsilon}^-\right\} \\
&\quad + \sum_{i=0}^{\infty} \int_{0+}^{\varepsilon} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} \in dx\}\mathbb{P}^0\left\{\gamma \le \nu_{2\varepsilon}^+ \wedge \nu_{-2\varepsilon}^-\right\} \\
&\quad + \sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} = 0\}\mathbb{P}^0\left\{\gamma \le \nu_{-\varepsilon}^- \wedge \nu_\varepsilon^+\right\} \\
&\quad + \sum_{i=0}^{\infty} \int_{0+}^{\varepsilon} \mathbb{P}_0\{T_i^+ < \nu_\varepsilon^+, X_{T_i^+} \in dx\}\mathbb{P}^0\left\{\gamma \le \nu_{-2\varepsilon}^- \wedge \nu_{2\varepsilon}^+\right\} \\
&\le 2\sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+\}\mathbb{P}^0\left\{\gamma \le \nu_{-2\varepsilon}^- \wedge \nu_{2\varepsilon}^+\right\} \\
&\le \frac{2}{\gamma}\sum_{i=0}^{\infty} \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+\}\mathbb{E}^0\left[\nu_{2\varepsilon}^+ \wedge \nu_{-2\varepsilon}^-\right] \\
&\le \frac{2}{\gamma}\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \mathbb{E}^0\left[\nu_{2\varepsilon}^+ \wedge \nu_{-2\varepsilon}^-\right] \\
&= O(\varepsilon^2),
\end{aligned}$$

where for the first inequality we have used spatial homogeneity for X and the fact $\gamma \leq \tau_\gamma$; for the fourth inequality we have used the strong Markov property, the spatial homogeneity and (2) to obtain

$$\mathbb{P}^0\left\{T_i^+ < \nu_\varepsilon^+\right\} \leq \mathbb{P}^0\left\{T_{i-1}^+ < \nu_\varepsilon^+\right\}\mathbb{P}^0\left\{\nu_{-\varepsilon}^- < \nu_\varepsilon^+\right\} < \frac{2}{3}\mathbb{P}^0\left\{T_{i-1}^+ < \nu_\varepsilon^+\right\},$$

and (3) was used for the last inequality. Therefore, (4) holds.

□

Lemma 2.3. *For $\sigma > 0$*

$$\mathbb{E}^0[\nu_\varepsilon^+ - \nu_\varepsilon^*; \nu_\varepsilon^+ < \infty] = O(\varepsilon^2),$$

where

$$\nu_\varepsilon^* := \sup\{t < \nu_\varepsilon^+ : X_t \leq 0\}.$$

Proof. Similarly to the proof for Lemma 2.2,

$$\begin{aligned}
\mathbb{E}^0[\nu_\varepsilon^+ - \nu_\varepsilon^*; \nu_\varepsilon^+ < \infty] &= \sum_{i=0}^\infty \mathbb{E}^0\left[\nu_\varepsilon^+ - \nu_\varepsilon^*; T_i^+ < \nu_\varepsilon^+ < T_i^-\right] \\
&= \sum_{i=0}^\infty \mathbb{E}^0\left[\mathbb{E}^{X_{T_i^+}}\left[\nu_\varepsilon^+ - \nu_\varepsilon^*; \nu_\varepsilon^+ < \nu_{-\varepsilon}^-\right]; T_i^+ < \nu_\varepsilon^+\right] \\
&\leq 2\sum_{i=0}^\infty \mathbb{P}^0\{T_i^+ < \nu_\varepsilon^+\}\mathbb{E}^0\left[\nu_{2\varepsilon}^+ \wedge \nu_{-2\varepsilon}^-\right] \\
&= O(\varepsilon^2).
\end{aligned}$$

□

3. The Laplace transform of $e^{-rT}X_T$ under $\tau_\gamma < T$ w.r.t. T

Let e_α be an exponential random variable with rate α, independent of everything else. As in [10], the goal is to find the Laplace transform

$$L_{\alpha,\gamma}(x, dy) := \frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha}; \tau_\gamma < e_\alpha, X_{e_\alpha} \in dy], \ \forall x, y,$$

in terms of functions with known Laplace transforms. In [10], $L_{\alpha,\gamma}(x, dy)$ is evaluated by conditioning on the stopping time τ_γ, where the distribution of τ_γ and W_{τ_γ} was provided (for τ_γ at least the Laplace transform was provided). Further it was shown that τ_γ and W_{τ_γ} are independent. For our purposes, we will instead condition on ν_0^+, i.e. the first time the process hits the boundary. This will give an equality for $L_{\alpha,\gamma}(x, dy)$ in terms of $L_{\alpha,\gamma}(0, dy)$,

$$I_{\alpha,\gamma}(dy) := \int_0^\infty dz\, \eta_1 e^{-\eta_1 z} L_{\alpha,\gamma}(z, dy)$$

and some terms whose Laplace transforms with respect to γ are known (these Laplace transforms will be provided in Sections 5 and 6). If $x < 0$, then we have that $\nu_0^+ < \tau_\gamma$ and we get with the strong Markov property of X_t and the lack-of-memory property of e_α

$$\begin{aligned}
L_{\alpha,\gamma}(x,dy) &= \frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy]\\
&= \frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy, X_{\nu_0^+}=0]\\
&\quad+\frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy, X_{\nu_0^+}>0]\\
&= \mathbb{E}^x[e^{-(r+\alpha)\nu_0^+};X_{\nu_0^+}=0]\,\frac{1}{\alpha}\mathbb{E}^0[e^{-re_\alpha};\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy]\\
&\quad+\int_0^\infty \mathbb{E}^x[e^{-(r+\alpha)\nu_0^+};X_{\nu_0^+}\in dz]\,\frac{1}{\alpha}\mathbb{E}^z[e^{-re_\alpha};\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy]\\
&= \Gamma_1(r+\alpha,-x)L_{\alpha,\gamma}(0,dy)+\Gamma_2(r+\alpha,-x)\int_0^\infty dz\,\eta_1 e^{-\eta_1 z}L_{\alpha,\gamma}(z,dy)\\
&= \Gamma_1(r+\alpha,-x)L_{\alpha,\gamma}(0,dy)+\Gamma_2(r+\alpha,-x)I_{\alpha,\gamma}(dy). \qquad (5)
\end{aligned}$$

Define $A_1(\alpha,\gamma,x,y)$ by

$$A_1(\alpha,\gamma,x,dy):=\mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<\nu_0^-\wedge e_\alpha, X_{e_\alpha}\in dy]/\alpha.$$

Then we get for $x>0$

$$\begin{aligned}
L_{\alpha,\gamma}(x,dy) &= \frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<\nu_0^-\wedge e_\alpha, X_{e_\alpha}\in dy]\\
&\quad+\frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\nu_0^-<\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy]\\
&= A_1(\alpha,\gamma,x,dy)+\frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\nu_0^-<\gamma,\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy, X_{\nu_0^-}=0]\\
&\quad+\frac{1}{\alpha}\mathbb{E}^x[e^{-re_\alpha};\nu_0^-<\gamma,\tau_\gamma<e_\alpha, X_{e_\alpha}\in dy, X_{\nu_0^-}<0]\\
&= A_1(\alpha,\gamma,x,dy)+\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}=0\right]L_{\alpha,\gamma}(0,dy)\\
&\quad+\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}<0\right]\int_0^\infty dz\,\eta_2 e^{-\eta_2 z}L_{\alpha,\gamma}(-z,dy).
\end{aligned}$$

By using (5),

$$\begin{aligned}
L_{\alpha,\gamma}(x,dy) &= A_1(\alpha,\gamma,x,dy)+\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}=0\right]L_{\alpha,\gamma}(0,dy)\\
&\quad+\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}<0\right]\left(\widehat{\Gamma}_1(r+\alpha,\eta_2)L_{\alpha,\gamma}(0,dy)+\widehat{\Gamma}_2(r+\alpha,\eta_2)I_{\alpha,\gamma}(dy)\right). \qquad (6)
\end{aligned}$$

To get expressions for $L_{\alpha,\gamma}(0,dy)$ and $I_{\alpha,\gamma}(dy)$ we search for two equalities that – besides these two functions – only involve functions with known Laplace transform. For the first equality we multiply (6) with $\eta_1 e^{-\eta_1 x}$ and integrate with respect to x. This leads to

$$\begin{aligned}
I_{\alpha,\gamma}(dy) &= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} A_1(\alpha,\gamma,x,dy) \\
&\quad + \left(\int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} = 0 \right] \right) L_{\alpha,\gamma}(0,dy) \\
&\quad + \left(\int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0 \right] \right) \\
&\qquad \left(\widehat{\Gamma}_1(r+\alpha,\eta_2) L_{\alpha,\gamma}(0,dy) + \widehat{\Gamma}_2(r+\alpha,\eta_2) I_{\alpha,\gamma}(dy) \right)
\end{aligned} \tag{7}$$

or

$$\begin{aligned}
&\left(1 - \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0 \right] \widehat{\Gamma}_2(r+\alpha,\eta_2) \right) I_{\alpha,\gamma}(dy) \\
&= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} A_1(\alpha,\gamma,x,dy) + \left\{ \int_0^\infty dx \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} = 0 \right] \right. \\
&\quad \left. + \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0 \right] \widehat{\Gamma}_1(r+\alpha,\eta_2) \right\} L_{\alpha,\gamma}(0,dy).
\end{aligned}$$

To get a second equation connecting $I_{\alpha,\gamma}(dy)$ and $L_{\alpha,\gamma}(0,dy)$, we will consider the Laplace transforms for (defective) distributions of $\{L_{\alpha,\gamma}(x,dy), y \geq 0\}$, $\{L_{\alpha,\gamma}(x,dy), y < 0\}$, $\{I_{\alpha,\gamma}(dy), y \geq 0\}$, $\{I_{\alpha,\gamma}(dy), y < 0\}$, $\{A_1(\alpha,\gamma,\varepsilon,dy), y \geq 0\}$ and $\{A_1(\alpha,\gamma,\varepsilon,dy), y < 0\}$. Put

$$\begin{aligned}
\hat{L}^+_{\alpha,\gamma,x}(\theta) &:= \int_0^\infty e^{-\theta y} L_{\alpha,\gamma}(x,dy),\ \theta \geq 0, \\
\hat{I}^+_{\alpha,\gamma}(\theta) &:= \int_0^\infty e^{-\theta y} I_{\alpha,\gamma}(dy),\ \theta \geq 0, \\
\hat{A}^+_1(\alpha,\gamma,\varepsilon,\theta) &:= \int_0^\infty e^{-\theta y} A_1(\alpha,\gamma,\varepsilon,dy),\ \theta \geq 0
\end{aligned}$$

and

$$\begin{aligned}
\hat{L}^-_{\alpha,\gamma,x}(\theta) &:= \int_{-\infty}^0 e^{\theta y} L_{\alpha,\gamma}(x,dy),\ \theta \geq 0, \\
\hat{I}^-_{\alpha,\gamma}(\theta) &:= \int_{-\infty}^0 e^{\theta y} I_{\alpha,\gamma}(dy),\ \theta \geq 0, \\
\hat{A}^-_1(\alpha,\gamma,\varepsilon,\theta) &:= \int_{-\infty}^0 e^{-\theta y} A_1(\alpha,\gamma,\varepsilon,dy),\ \theta \geq 0.
\end{aligned}$$

We now consider the derivative of $\hat{L}^{+}_{\alpha,\gamma,x}(\theta)$ with respect to x at $x=0$. Note that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\frac{\hat{L}^{+}_{\alpha,\gamma,\varepsilon}(\theta)-\hat{L}^{+}_{\alpha,\gamma,0}(\theta)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0^+}\Bigg\{\frac{\hat{A}^{+}_{1}(\alpha,\gamma,\varepsilon,\theta)}{\varepsilon}-\frac{1-E^{\varepsilon}\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}=0\right]}{\varepsilon}\hat{L}^{+}_{\alpha,\gamma,0}(\theta)\\
&\qquad+\frac{E^{\varepsilon}\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}<0\right]}{\varepsilon}\Big(\widehat{\Gamma}_1(r+\alpha,\eta_2)\hat{L}^{+}_{\alpha,\gamma,0}(\theta)+\widehat{\Gamma}_2(r+\alpha,\eta_2)\hat{I}^{+}_{\alpha,\gamma}(\theta)\Big)\Bigg\}.
\end{aligned}\tag{8}$$

On the other hand, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\frac{\hat{L}^{+}_{\alpha,\gamma,0}(\theta)-\hat{L}^{+}_{\alpha,\gamma,-\varepsilon}(\theta)}{\varepsilon}&=\lim_{\varepsilon\to 0^+}\left\{\frac{1-\Gamma_1(r+\alpha,\varepsilon)}{\varepsilon}\hat{L}^{+}_{\alpha,\gamma,0}(\theta)-\frac{\Gamma_2(r+\alpha,\varepsilon)}{\varepsilon}\hat{I}^{+}_{\alpha,\gamma}(\theta)\right\}\\
&=\Lambda_1(r+\alpha)\hat{L}^{+}_{\alpha,\gamma,0}(\theta)-\Lambda_2(r+\alpha)\hat{I}^{+}_{\alpha,\gamma}(\theta).
\end{aligned}\tag{9}$$

If $\hat{L}^{+}_{\alpha,\gamma,x}(\theta)$ is differentiable in $x=0$, then the left sides of (8) and (9) are equal and we get an equation on $\hat{L}^{+}_{\alpha,\gamma,0}(\theta)$ and $\hat{I}^{+}_{\alpha,\gamma}(\theta)$. Similarly, an equation on $\hat{L}^{-}_{\alpha,\gamma,0}(\theta)$ and $\hat{I}^{-}_{\alpha,\gamma}(\theta)$ follows. As a result, we obtain a second equality for $I_{\gamma,\alpha}(dy)$ and $L_{\alpha,\gamma}(0,dy)$.

The following lemma shows that $\hat{L}^{+}_{\alpha,\gamma,x}(\theta)$ is differentiable in $x=0$. For $\varepsilon>0$, consider the exit time of X from level ε. Then we have

Lemma 3.1. *Under the above conditions and for $\varepsilon>0$ one has*

$$\hat{L}^{+}_{\alpha,\gamma,0}(\theta)=\Gamma_1(r+\alpha,\varepsilon)\hat{L}^{+}_{\alpha,\gamma,\varepsilon}(\theta)+\Gamma_2(r+\alpha,\varepsilon)\hat{I}^{+}_{\alpha,\gamma}(\theta)+o(\varepsilon)$$

and

$$\hat{L}^{-}_{\alpha,\gamma,0}(\theta)=\Gamma_1(r+\alpha,\varepsilon)\hat{L}^{-}_{\alpha,\gamma,\varepsilon}(\theta)+\Gamma_2(r+\alpha,\varepsilon)\hat{I}^{-}_{\alpha,\gamma}(\theta)+o(\varepsilon).\tag{10}$$

Proof. Recall that

$$\nu_\varepsilon^* := \sup\{t\le \nu_\varepsilon^+ : X_t\le 0\}.$$

Conditioning on the first time that the process crosses the level ε and comparing τ_γ with ν_ε^+ and $\nu_\varepsilon^+ + \nu_0^-\circ\nu_\varepsilon^+$ we get:

$$\begin{aligned}
&\hat{L}_{\alpha,\gamma,0}(\theta)\\
&=\int_0^\infty dt\,e^{-\alpha t}\int_0^\infty e^{-\theta y}\,\mathbb{E}^0\left[e^{-rt};\tau_\gamma<t,X_t\in dy\right]\\
&=\int_0^\infty dt\,e^{-(\alpha+r)t}\int_0^\infty e^{-\theta y}\,\mathbb{P}^0\left\{\tau_\gamma<t,X_t\in dy\right\}\\
&=\int_0^\infty dt\,e^{-(\alpha+r)t}\int_0^\infty e^{-\theta y}\,\mathbb{P}^0\left\{\tau_\gamma\le \nu_\varepsilon^+\wedge t,X_t\in dy\right\}\\
&\quad+\int_0^\infty dt\,e^{-(r+\alpha)t}\int_0^t\int_0^\infty e^{-\theta y}\,\mathbb{P}^0\left\{\nu_\varepsilon^+\in dl, X_{\nu_\varepsilon^+}=\varepsilon, l<\tau_\gamma<t, X_t\in dy\right\}\\
&\quad+\int_0^\infty dt\,e^{-(r+\alpha)t}\int_0^t\int_{\varepsilon+}^\infty\int_0^\infty e^{-\theta y}\,\mathbb{P}^0\left\{\nu_\varepsilon^+\in dl, X_{\nu_\varepsilon^+}\in dz, l<\tau_\gamma<t, X_t\in dy\right\}
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty dt\, e^{-(\alpha+r)t}\,\mathbb{P}^0\left\{\tau_\gamma \le \nu_\varepsilon^+ \wedge t\right\} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\left\{X_t \in dy | \tau_\gamma \le \nu_\varepsilon^+ \wedge t\right\} \\
&\quad + \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma < t, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma, X_t \in dy\} \\
&\quad + \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma < t, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < t - \nu_\varepsilon^+ < \gamma \le \nu_0^- \circ \theta_{\nu_\varepsilon^+}, X_t \in dy\} \\
&\quad - \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma < t, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma, \\
&\qquad\qquad \nu_\varepsilon^+ + \tau_\gamma \circ \theta_{\nu_\varepsilon^+} < t, X_t \in dy\} \\
&\quad + \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^t \mathbb{P}^0\left\{\nu_\varepsilon^+ \in dl, X_{\nu_\varepsilon^+} = \varepsilon\right\} \int_0^\infty e^{-\theta y}\,\mathbb{P}^\varepsilon\left\{\tau_\gamma < t - l, X_{t-l} \in dy\right\} \\
&\quad + \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^t \int_{\varepsilon+}^\infty \mathbb{P}^0\left\{\nu_\varepsilon^+ \in dl, X_{\nu_\varepsilon^+} \in dz\right\} \int_0^\infty e^{-\theta y}\,\mathbb{P}^z\left\{\tau_\gamma < t - l, X_{t-l} \in dy\right\}
\end{aligned}$$

where we take Laplace transforms on convolutions of functions for the fourth equality. By Lemma 2.3 and $\mathbb{P}^0\{X_{\nu_\varepsilon^+} > \varepsilon, \nu_\varepsilon^+ < \infty\} = O(\varepsilon)$, we have $\mathbb{P}^\varepsilon\{\nu_0^- > \gamma - \varepsilon^{2/3}\} = O(\varepsilon)$. Further

$$\begin{aligned}
&(r+\alpha)\int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma < t, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma, X_{e_\alpha} \in dy\} \\
&\le \mathbb{P}^0\{\gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma\} \\
&\le \mathbb{P}^0\{\nu_\varepsilon^+ < \infty, \nu_\varepsilon^+ - \nu_\varepsilon^* > \varepsilon^{2/3}\} + \mathbb{P}^0\{\gamma - \varepsilon^{2/3} < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma\} \\
&\le O(\varepsilon^{4/3}) + \mathbb{P}^0\{\nu_\varepsilon^+ < \infty, \nu_0^- \circ \theta_{\nu_\varepsilon^+} > \gamma - \varepsilon^{2/3}\} - \mathbb{P}^0\{\nu_\varepsilon^+ < \infty, \nu_0^- \circ \theta_{\nu_\varepsilon^+} \ge \gamma\} \\
&= O(\varepsilon^{4/3}) + \mathbb{E}^0\left[\mathbb{P}^{X_{\nu_\varepsilon^+}}\{\nu_0^- > \gamma - \varepsilon^{2/3}\} - \mathbb{P}^{X_{\nu_\varepsilon^+}}\{\nu_0^- \ge \gamma\}; \nu_\varepsilon^+ < \infty\right] \\
&= O(\varepsilon^{4/3}) + \mathbb{E}^0\left[\mathbb{P}^{X_{\nu_\varepsilon^+}}\{\nu_0^- > \gamma - \varepsilon^{2/3}\}\left(1 - \mathbb{P}^{X_{\nu_\varepsilon^+}}\{\nu_0^- \ge \gamma | \nu_0^- > \gamma - \varepsilon^{2/3}\}\right); \nu_\varepsilon^+ < \infty\right] \\
&= o(\varepsilon).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma < t, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma, \\
&\qquad\qquad \nu_\varepsilon^+ + \tau_\gamma \circ \theta_{\nu_\varepsilon^+} < t, X_t \in dy\} \\
&\le \mathbb{P}^0\{\gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < \nu_0^- \circ \theta_{\nu_\varepsilon^+} < \gamma\} \\
&= o(\varepsilon)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha \int_0^\infty dt\, e^{-(r+\alpha)t} \int_0^\infty e^{-\theta y}\,\mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < t - \nu_\varepsilon^+ < \gamma \le \nu_0^- \circ \theta_{\nu_\varepsilon^+}, X_t \in dy\} \\
&\le \mathbb{P}^0\{\nu_\varepsilon^+ < \tau_\gamma, \gamma - (\nu_\varepsilon^+ - \nu_\varepsilon^*) < e_\alpha - \nu_\varepsilon^+ < \gamma \le \nu_0^- \circ \theta_{\nu_\varepsilon^+}\} \\
&= o(\varepsilon).
\end{aligned}$$

It follows that

$$\begin{aligned}
\hat{L}_{\alpha,\gamma,0}(\theta) &= o(\varepsilon) + \mathbb{E}^0\left[e^{-(r+\alpha)\nu_\varepsilon^+}; X_{\nu_\varepsilon^+} = \varepsilon\right] \int_0^\infty e^{-\theta y}\, \mathbb{E}^\varepsilon[e^{-re_\alpha}; \tau_\gamma < e_\alpha, X_{e_\alpha} \in dy]/\alpha \\
&\quad + \int_{\varepsilon+}^\infty \mathbb{E}^0\left[e^{-(r+\alpha)\nu_\varepsilon^+}; X_{\nu_\varepsilon^+} \in dz\right] \int_0^\infty e^{-\theta y}\, \mathbb{E}^z[e^{-re_\alpha}; \tau_\gamma < e_\alpha, X_{e_\alpha} \in dy]/\alpha \\
&= o(\varepsilon) + \Gamma_1(r+\alpha,\varepsilon)\hat{L}^+_{\alpha,\gamma,\varepsilon}(\theta) + \Gamma_2(r+\alpha,\varepsilon)\int_\varepsilon^\infty dx\, \eta_1 e^{-\eta_1(x-\varepsilon)} \int_0^\infty e^{-\theta y} L_{\alpha,\gamma}(x,dy) \\
&= o(\varepsilon) + \Gamma_1(r+\alpha,\varepsilon)\hat{L}^+_{\alpha,\gamma,\varepsilon}(\theta) + \Gamma_2(r+\alpha,\varepsilon)\hat{I}^+_{\alpha,\gamma}(\theta).
\end{aligned}$$

Equation (10) can be obtained similarly. □

To put everything together, define:

$$\begin{aligned}
A_2(\alpha,\gamma,dy) &:= \lim_{\varepsilon\to 0^+} \frac{A_1(\alpha,\gamma,\varepsilon,dy)}{\varepsilon}, \\
B_1(\alpha,\gamma) &:= \Lambda_2(r+\alpha) + \lim_{\varepsilon\to 0^+} \frac{\mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0\right]}{\varepsilon} \widehat{\Gamma}_2(r+\alpha,\eta_2), \\
B_2(\alpha,\gamma) &:= \Lambda_1(r+\alpha) + \lim_{\varepsilon\to 0^+} \frac{1 - \mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} = 0\right]}{\varepsilon} \\
&\quad - \lim_{\varepsilon\to 0^+} \frac{\mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0\right]}{\varepsilon} \widehat{\Gamma}_1(r+\alpha,\eta_2).
\end{aligned}$$

The existence of A_2 follows from the existence of $\lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}^\varepsilon\left[1; \gamma < \nu_0^-\right]$.
Now equation (8) and (9) can be combined to

$$B_2(\alpha,\gamma)L_{\alpha,\gamma}(0,dy) = A_2(\alpha,\gamma,dy) + B_1(\alpha,\gamma)I_{\alpha,\gamma}(dy).$$

With

$$\begin{aligned}
B_3(\alpha,\gamma) &:= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} = 0\right] \\
&\quad + \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0\right] \widehat{\Gamma}_1(r+\alpha,\eta_2), \\
B_4(\alpha,\gamma) &:= 1 - \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-} < 0\right] \widehat{\Gamma}_2(r+\alpha,\eta_2), \\
A_3(\alpha,\gamma,dy) &:= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} A_1(\alpha,\gamma,x,dy),
\end{aligned}$$

equation (7) reads

$$B_4(\alpha,\gamma)I_{\alpha,\gamma}(dy) = A_3(\alpha,\gamma,dy) + B_3(\alpha,\gamma)L_{\alpha,\gamma}(0,dy).$$

It follows that

$$I_{\alpha,\gamma}(dy) = \frac{B_2(\alpha,\gamma)A_3(\alpha,\gamma,dy) + A_2(\alpha,\gamma,dy)B_3(\alpha,\gamma)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma) - B_1(\alpha,\gamma)B_3(\alpha,\gamma)}, \tag{11}$$

$$L_{\alpha,\gamma}(0,dy) = \frac{B_1(\alpha,\gamma)A_3(\alpha,\gamma,dy) + A_2(\alpha,\gamma,dy)B_4(\alpha,\gamma)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma) - B_1(\alpha,\gamma)B_3(\alpha,\gamma)}. \tag{12}$$

Finally with

$$\begin{aligned}
B_5(\alpha,\gamma,x) &:= \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}=0\right],\\
B_6(\alpha,\gamma,x) &:= \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma, X_{\nu_0^-}<0\right],
\end{aligned}$$

we get for $x>0$

$$\begin{aligned}
L_{\alpha,\gamma}(x,dy) = A_1(\alpha,\gamma,x,dy) + B_5(\alpha,\gamma,x)L_{\alpha,\gamma}(0,dy)+&\\
B_6(\alpha,\gamma,x)\left[\widehat{\Gamma}_1(r+\alpha,\eta_2)L_{\alpha,\gamma}(0,dy)+\widehat{\Gamma}_2(r+\alpha,\eta_2)I_{\alpha,\gamma}(dy)\right]&
\end{aligned} \tag{13}$$

and for $x<0$,

$$L_{\alpha,\gamma}(x,dy) = \Gamma_1(r+\alpha,-x)L_{\alpha,\gamma}(0,dy) + \Gamma_2(r+\alpha,-x)I_{\alpha,\gamma}(dy). \tag{14}$$

4. The evaluation of the option price

The Laplace transform of the option price with respect to the maturity time T is given by

$$\begin{aligned}
\widehat{C}(\gamma,k,S_0,\alpha) &:= \int_0^\infty dT\ e^{-\alpha T}C(\gamma,k,S_0,T)\\
&= \int_0^\infty dT\ e^{-(r+\alpha)T}\,\mathbb{E}^{\log(S_0)}[(e^{X_T}-k)^+;\tau_\gamma<T]\\
&= \frac{1}{\alpha}\mathbb{E}^{\log(S_0)}[e^{-re_\alpha}(e^{X_{e_\alpha}}-k)^+;\tau_\gamma<e_\alpha]\\
&= \int_{-\infty}^\infty L_{\alpha,\gamma}(\log(S_0),dy)\ (e^x-k)^+.
\end{aligned}$$

We can also interpret $\widehat{C}(\gamma,k,S_0,\alpha)$ as the option price with respect to an independent exponential maturity time.
Denote with

$$\widetilde{A}_1(\alpha,\gamma,x,k) := \int_{-\infty}^\infty A_1(\alpha,\gamma,x,dy)\ (e^y-k)^+,$$

and for $i=2,3$

$$\widetilde{A}_i(\alpha,\gamma,k) := \int_{-\infty}^\infty A_i(\alpha,\gamma,dy)\ (e^y-k)^+,$$

then with the equations (11), (12) and (14) we get for $S_0\geq 1$

$$\begin{aligned}\widehat{C}(\gamma,k,S_0,\alpha) =\, &\widetilde{A}_1(\alpha,\gamma,\log(S_0),k)\\ &+\Big(B_5(\alpha,\gamma,\log(S_0))+B_6(\alpha,\gamma,\log(S_0))\widehat{\Gamma}_1(r+\alpha,\eta_2)\Big)\\ &\quad\times\frac{B_1(\alpha,\gamma)\widetilde{A}_3(\alpha,\gamma,k)+B_4(\alpha,\gamma)\widetilde{A}_2(\alpha,\gamma,k)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma)-B_1(\alpha,\gamma)B_3(\alpha,\gamma)}\\ &+B_6(\alpha,\gamma,\log(S_0))\widehat{\Gamma}_2(r+\alpha,\eta_2)\\ &\quad\times\frac{B_2(\alpha,\gamma)\widetilde{A}_3(\alpha,\gamma,k)+B_3(\alpha,\gamma)\widetilde{A}_2(\alpha,\gamma,k)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma)-B_1(\alpha,\gamma)B_3(\alpha,\gamma)},\end{aligned}\tag{15}$$

and for $S_0 \leq 1$

$$\begin{aligned}\widehat{C}(\gamma,k,S_0,\alpha) = \,&\Gamma_1(r+\alpha,-\log(S_0))\frac{B_1(\alpha,\gamma)\widetilde{A}_3(\alpha,\gamma,k)+B_4(\alpha,\gamma)\widetilde{A}_2(\alpha,\gamma,k)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma)-B_1(\alpha,\gamma)B_3(\alpha,\gamma)}\\ &+\Gamma_2(r+\alpha,-\log(S_0))\frac{B_2(\alpha,\gamma)\widetilde{A}_3(\alpha,\gamma,k)+B_3(\alpha,\gamma)\widetilde{A}_2(\alpha,\gamma,k)}{B_2(\alpha,\gamma)B_4(\alpha,\gamma)-B_1(\alpha,\gamma)B_3(\alpha,\gamma)}.\end{aligned}$$

Explicit expressions for the Laplace transform of the functions $B_i(\cdot)$ are provided in Section 5 and for the Laplace transform of the functions $\widetilde{A}_i(\cdot)$ in Section 6.

Remark 4.1. In Section 6 it will be shown that the $\widetilde{A}_i$ only exist if the positive roots of $r+\alpha = G(\theta)$ (to be defined there) are larger than 1. Otherwise, we can use a new rate r_1 such that for all considered α, all roots of $r_1+\alpha = G(\theta)$ are larger than 1. One can then evaluate the option price with respect to r_1 and afterwards correct the error through multiplying by $e^{(r_1-r)T}$. □

Remark 4.2. For a barrier L other than $L = 1$, one has

$$C(\gamma,k,S_0,T,L) = L\; C(\gamma,k/L,S_0/L,T,1).$$

□

Remark 4.3. If we want to evaluate the price of a Parisian option at the time t and under the condition that the underlying process is already above the barrier for a duration of γ_t , then we only need to change $\widetilde{A}_1(\alpha,\gamma,x,k)$ to $\widetilde{A}_1(\alpha,\gamma-\gamma_t,X_t,k)$, $B_5(\alpha,\gamma,x)$ to $B_5(\alpha,\gamma-\gamma_t,X_t)$ and $B_6(\alpha,\gamma,x)$ to $B_6(\alpha,\gamma-\gamma_t,X_t)$ in the corresponding formula. □

Remark 4.4. Note that due to $\gamma < T$ the inverse Laplace transform of $\widetilde{A}_1(\alpha,\gamma,x,k)$ is

$$\begin{aligned}&e^{-rT}\mathbb{E}[(e^{X_T}-k)^+,\tau_\gamma<\nu_0^-\wedge T]\\ &\qquad= e^{-rT}\mathbb{E}[(e^{X_T}-k)^+,\gamma<\nu_0^-\wedge T] = e^{-rT}\mathbb{E}[(e^{X_T}-k)^+,T\wedge\gamma<\nu_0^-].\end{aligned}$$

Hence we can replace $\widetilde{A}_1(\alpha,\gamma,x,k)$ by $\mathbb{E}^x[e^{-re_\alpha}(e^{X_{e_\alpha}}-k)^+; e_\alpha\wedge\tau_\gamma<\nu_0^-]/\alpha$, which will lead to a simpler expression for numerical purposes (cf. Section 6). □

5. The Laplace transform of the functions $B_i(\cdot)$

To evaluate B_i note that

$$
\begin{aligned}
&\int_0^\infty d\gamma\, \beta e^{-\beta\gamma} \lim_{\varepsilon\to 0^+} \frac{1-\mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-}=0\right]}{\varepsilon} \\
&\quad = \lim_{\varepsilon\to 0^+} \frac{1-\int_0^\infty d\gamma\, \beta e^{-\beta\gamma}\mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-}=0\right]}{\varepsilon} \\
&\quad = \lim_{\varepsilon\to 0^+} \frac{1-\mathbb{E}^\varepsilon\left[\int_0^\infty d\gamma\, \beta e^{-\beta\gamma} I_{\{\nu_0^-<\gamma\}} e^{-(r+\alpha)\nu_0^-}; X_{\nu_0^-}=0\right]}{\varepsilon} \qquad (16)\\
&\quad = \lim_{\varepsilon\to 0^+} \frac{1-\mathbb{E}^\varepsilon\left[e^{-(r+\alpha+\beta)\nu_0^-}; X_{\nu_0^-}=0\right]}{\varepsilon} \\
&\quad = \lim_{\varepsilon\to 0^+} \frac{1-\Gamma_1^*(r+\alpha+\beta,\epsilon)}{\varepsilon} = \Lambda_1^*(r+\alpha+\beta)
\end{aligned}
$$

and similarly

$$
\int_0^\infty d\gamma\, \beta e^{-\beta\gamma} \lim_{\varepsilon\to 0^+} \frac{\mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma, X_{\nu_0^-}<0\right]}{\varepsilon} = \Lambda_2^*(r+\alpha+\beta).
$$

It follows that the Laplace transforms of B_1 and B_2 with respect to γ can be written as

$$
\begin{aligned}
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_1(\alpha,\gamma) &= \Lambda_2(r+\alpha) + \widehat{\Gamma}_2(r+\alpha,\eta_2)\Lambda_2^*(r+\alpha+\beta), \\
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_2(\alpha,\gamma) &= \Lambda_1(r+\alpha) + \Lambda_1^*(r+\alpha+\beta) - \widehat{\Gamma}_1(r+\alpha,\eta_2)\Lambda_2^*(r+\alpha+\beta).
\end{aligned}
$$

Similarly to (16), we get the Laplace transforms of B_3, B_4, B_5 and B_6 with respect to γ:

$$
\begin{aligned}
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_3(\alpha,\gamma) &= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \Gamma_1^*(r+\alpha+\beta,x) \\
&\quad + \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \Gamma_2^*(r+\alpha+\beta,x) \int_0^\infty du\, \eta_2 e^{-\eta_2 u}\, \Gamma_1(r+\alpha,u) \\
&= \widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1) + \widehat{\Gamma}_2^*(r+\alpha+\beta,\eta_1)\, \widehat{\Gamma}_1(r+\alpha,\eta_2), \\
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_4(\alpha,\gamma) &= 1 - \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, \Gamma_2^*(r+\alpha+\beta,x) \int_0^\infty du\, \eta_2 e^{-\eta_2 u}\, \Gamma_2(r+\alpha,u) \\
&= 1 - \widehat{\Gamma}_2^*(r+\alpha+\beta,\eta_1)\, \widehat{\Gamma}_2(r+\alpha,\eta_2), \\
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_5(\alpha,\gamma) &= \Gamma_1^*(r+\alpha+\beta,x), \\
\beta\int_0^\infty d\gamma\, e^{-\beta\gamma} B_6(\alpha,\gamma) &= \Gamma_2^*(r+\alpha+\beta,x).
\end{aligned}
$$

6. The Laplace transform of the functions $A_i(\cdot)$

In this section we provide explicit expressions for the Laplace transforms of $A_1(\alpha,\gamma,x,y)$, $A_2(\alpha,\gamma,y)$ and $A_3(\alpha,\gamma,y)$ as well as for $\widetilde{A}_1(\alpha,\gamma,x,k)$, $\widetilde{A}_2(\alpha,\gamma,k)$ and $\widetilde{A}_3(\alpha,\gamma,k)$. Put

$$p_\alpha(x) := \frac{1}{\alpha}\mathbb{P}^0\{X_{e_\alpha}\in dx\}/dx.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, e^{i\theta x}\alpha p_\alpha(x) &= \mathbb{E}^0[e^{i\theta X_{e_\alpha}}]\\ &= \alpha\int_0^\infty dt\, e^{(G(i\theta)-\alpha)t} = \frac{\alpha}{\alpha - G(i\theta)}\end{aligned} \tag{17}$$

for $|\theta|$ either small or large enough such that the real part of $G(i\theta)-\alpha$ is negative. With a partial fraction decomposition we get that

$$\begin{aligned}p_\alpha(x) &= \frac{2}{\sigma^2}\left(\frac{(\eta_1-\beta_1)(\eta_2+\beta_1)}{(\beta_2-\beta_1)(\beta_3+\beta_1)(\beta_4+\beta_1)}\,e^{-\beta_1 x} + \frac{(\eta_1-\beta_2)(\eta_2+\beta_2)}{(\beta_1-\beta_2)(\beta_3+\beta_2)(\beta_4+\beta_2)}\,e^{-\beta_2 x}\right)1_{\{x\ge 0\}}\\ &\quad + \frac{2}{\sigma^2}\left(\frac{(\eta_1+\beta_3)(\eta_2-\beta_3)}{(\beta_1+\beta_3)(\beta_2+\beta_3)(\beta_4-\beta_3)}\,e^{\beta_3 x} + \frac{(\eta_1+\beta_4)(\eta_2-\beta_4)}{(\beta_1+\beta_4)(\beta_2+\beta_4)(\beta_3-\beta_4)}\,e^{\beta_4 x}\right)1_{\{x\le 0\}}.\end{aligned}$$

Proposition 6.1. *For $x>0$ we have*

$$A_1(\alpha,\gamma,x,dy) = A_1(\alpha,\gamma,x,y)dy$$

where

$$\begin{aligned}\int_0^\infty d\gamma\, e^{-\beta\gamma}A_1(\alpha,\gamma,x,y) &= p_{r+\alpha+\beta}*p_{r+\alpha}(y-x) - \Gamma_1^*(r+\alpha+\beta,x)\,p_{r+\alpha+\beta}*p_{r+\alpha}(y)\\ &\quad - \Gamma_2^*(r+\alpha+\beta,x)\int_0^\infty du\,\eta_2 e^{-\eta_2 u}\,p_{r+\alpha+\beta}*p_{r+\alpha}(u+y)\end{aligned} \tag{18}$$

and $p_{r+\alpha+\beta}*p_{r+\alpha}$ denotes the convolution of the two functions.

Proof. At first, note that

$$\begin{aligned}A_1(\alpha,\gamma,x,dy) &:= \mathbb{E}^x[e^{-re_\alpha};\tau_\gamma<\nu_0^-\wedge e_\alpha, X_{e_\alpha}\in dy]/\alpha\\ &= \mathbb{E}^x[e^{-re_\alpha};\gamma<\nu_0^-\wedge e_\alpha, X_{e_\alpha}\in dy]/\alpha\\ &= \mathbb{E}^x[e^{-re_\alpha};\gamma<e_\alpha, X_{e_\alpha}\in dy]/\alpha\\ &\qquad - \mathbb{E}^x[e^{-re_\alpha};\nu_0^-<\gamma<e_\alpha, X_{e_\alpha}\in dy]/\alpha.\end{aligned} \tag{19}$$

For the Laplace transform of the first expectation we get with conditioning on X_γ

$$\begin{aligned}&\int_0^\infty d\gamma\, e^{-\beta\gamma}\int_\gamma^\infty dt\, e^{-\alpha t}\,\mathbb{E}^x[e^{-rt};X_t\in dy]\\ &= \int_0^\infty d\gamma\, e^{-(r+\alpha+\beta)\gamma}\int_{-\infty}^\infty \mathbb{P}^x\{X_\gamma\in du\}\int_0^\infty dt\, e^{-(r+\alpha)t}\,\mathbb{P}^u\{X_t\in dy\}\\ &= \int_{-\infty}^\infty du\, p_{r+\alpha+\beta}(u-x)p_{r+\alpha}(y-u)dy\\ &= \int_{-\infty}^\infty du\, p_{r+\alpha+\beta}(u)p_{r+\alpha}(y-x-u)dy\end{aligned}$$

and for the Laplace transform of the second expectation we get with conditioning on $X_{\nu_0^-}$

$$
\begin{aligned}
&\mathbb{E}^x[e^{-re_\alpha}; \nu_0^- < e_\beta < e_\alpha, X_{e_\alpha} \in dy]/\alpha\beta \\
&= \int_0^\infty d\gamma\, e^{-\beta\gamma} \int_\gamma^\infty dt\, e^{-\alpha t} e^{-rt}\, \mathbb{P}^x\{\nu_0^- < \gamma, X_t \in dy\} \\
&= \int_0^\infty d\gamma\, e^{-\beta\gamma} \int_\gamma^\infty dt\, e^{-(r+\alpha)t} \int_0^\gamma \mathbb{P}^x\{\nu_0^- \in dl, X_{\nu_0^-} = 0\} \int_{-\infty}^\infty \mathbb{P}^0\{X_{\gamma-l} \in dv\} \mathbb{P}^v\{X_{t-\gamma} \in dy\} \\
&\quad + \int_0^\infty d\gamma\, e^{-\beta\gamma} \int_\gamma^\infty dt\, e^{-(r+\alpha)t} \int_{-\infty}^0 \int_0^\gamma \mathbb{P}^x\{\nu_0^- \in dl, X_{\nu_0^-} \in du\} \\
&\qquad\qquad \times \int_{-\infty}^\infty \mathbb{P}^u\{X_{\gamma-l} \in dv\} \mathbb{P}^v\{X_{t-\gamma} \in dy\} \\
&= \int_0^\infty d\gamma\, e^{-(r+\alpha+\beta)\gamma} \int_0^\gamma \mathbb{P}^x\{\nu_0^- \in dl, X_{\nu_0^-} = 0\} \int_{-\infty}^\infty \mathbb{P}^0\{X_{\gamma-l} \in dv\} \int_0^\infty dt\, e^{-(r+\alpha)t}\, \mathbb{P}^v\{X_t \in dy\} \\
&\quad + \int_0^\infty d\gamma\, e^{-(r+\alpha+\beta)\gamma} \int_{-\infty}^0 \int_0^\gamma \mathbb{P}^x\{\nu_0^- \in dl, X_{\nu_0^-} \in du\} \int_{-\infty}^\infty \mathbb{P}^u\{X_{\gamma-l} \in dv\} \\
&\qquad\qquad \times \int_0^\infty dt\, e^{-(r+\alpha)t}\, \mathbb{P}^v\{X_t \in dy\} \\
&= \int_0^\infty e^{-(r+\alpha+\beta)\gamma}\, \mathbb{P}^x\{\nu_0^- \in d\gamma, X_{\nu_0^-} = 0\} \int_{-\infty}^\infty dv\, p_{r+\alpha+\beta}(v) p_{r+\alpha}(y-v) dy \\
&\quad + \int_{-\infty}^0 \int_0^\infty e^{-(r+\alpha+\beta)\gamma}\, \mathbb{P}^x\{\nu_0^- \in d\gamma, X_{\nu_0^-} \in du\} \int_{-\infty}^\infty dv\, p_{r+\alpha+\beta}(v-u) p_{r+\alpha}(y-v) dy,
\end{aligned}
$$

where for the last equality we apply integration by parts and the Laplace transform to convolutions of functions. □

We hence (after integration with respect to y) have obtained the Laplace transform w.r.t. γ of the first term in (15). However, this expression is quite involved and in view of Remark 4.4 we instead consider the following simpler Laplace transform that on the domain $\{\gamma < T\}$ corresponds to the same original function and hence leads to a simplification of the numerical procedure of Laplace inversion. Mimicking the proof of Proposition 6.1 we obtain

Proposition 6.2.

$$
\begin{aligned}
&\beta \int_0^\infty d\gamma\; e^{-\beta\gamma}\, \mathbb{E}^x[e^{-re_\alpha}; e_\alpha \wedge \tau_\gamma < \nu_0^-, X_{e_\alpha} \in dy]/\alpha \\
&= p_{r+\alpha}(y-x) - \Gamma_1^*(r+\alpha+\beta, x) p_{r+\alpha}(y) \\
&\quad - \Gamma_2^*(r+\alpha+\beta, x) \int_0^\infty du\; \eta_2 e^{-\eta_2 u} p_{r+\alpha}(u+y).
\end{aligned}
$$

Remark 6.1. Lemma 6.1 together with (6), (11) and (12) shows that $L_{\alpha,\gamma}(x, dy) = L_{\alpha,\gamma}(x, y)dy$ and $I_{\alpha,\gamma}(dy) = I_{\alpha,\gamma}(y)dy$. □

Remark 6.2. $p_{\alpha_1} * p_{\alpha_2}(x)$ is given by

$$\begin{aligned}p_{\alpha_1} * p_{\alpha_2}(x) = 1_{\{x\geq 0\}} &\left(A^{1,1}_{\alpha_1,\alpha_2} e^{-\beta_1^{\alpha_1} x} + A^{1,2}_{\alpha_1,\alpha_2} e^{-\beta_2^{\alpha_1} x} + A^{2,1}_{\alpha_1,\alpha_2} e^{-\beta_1^{\alpha_2} x} + A^{2,2}_{\alpha_1,\alpha_2} e^{-\beta_2^{\alpha_2} x}\right)\\ &+ 1_{\{x\leq 0\}} \left(A^{1,3}_{\alpha_1,\alpha_2} e^{\beta_3^{\alpha_1} x} + A^{1,4}_{\alpha_1,\alpha_2} e^{\beta_4^{\alpha_1} x} + A^{2,3}_{\alpha_1,\alpha_2} e^{\beta_3^{\alpha_2} x} + A^{2,4}_{\alpha_1,\alpha_2} e^{\beta_4^{\alpha_2} x}\right),\end{aligned}$$

where $\beta_i^{\alpha_i}$ denote the absolute values of the roots of $G(x) = \alpha_i$ (cf. (1)), for $j \in \{1,2\}$

$$A^{i,j}_{\alpha_1,\alpha_2} = \frac{2}{\sigma^2(\alpha_{3-i} - \alpha_i)} \frac{(\eta_1 - \beta_j^{\alpha_i})(\eta_2 + \beta_j^{\alpha_i})}{(\beta_{3-j}^{\alpha_i} - \beta_j^{\alpha_i})(\beta_3^{\alpha_i} + \beta_j^{\alpha_i})(\beta_4^{\alpha_i} + \beta_j^{\alpha_i})},$$

and for $j \in \{3,4\}$

$$A^{i,j}_{\alpha_1,\alpha_2} = \frac{2}{\sigma^2(\alpha_{3-i} - \alpha_i)} \frac{(\eta_1 + \beta_j^{\alpha_i})(\eta_2 - \beta_j^{\alpha_i})}{(\beta_1^{\alpha_i} + \beta_j^{\alpha_i})(\beta_2^{\alpha_i} + \beta_j^{\alpha_i})(\beta_{7-j}^{\alpha_i} - \beta_j^{\alpha_i})}.$$

□

It follows that

$$\begin{aligned}\int_0^\infty d\gamma\, e^{-\beta\gamma} A_2(\alpha,\gamma,y) &= \lim_{\varepsilon\to 0+} \Gamma_1(r+\alpha,\varepsilon) \int_0^\infty d\gamma\, e^{-\beta\gamma} A_1(\alpha,\gamma,\varepsilon,y)/\varepsilon\\ &= \lim_{\varepsilon\to 0+} \int_0^\infty d\gamma\, e^{-\beta\gamma} A_1(\alpha,\gamma,\varepsilon,y)/\varepsilon\\ &= -(p_{r+\alpha+\beta} * p_{r+\alpha})'(y) + \Lambda_1^*(r+\alpha+\beta)\, p_{r+\alpha+\beta} * p_{r+\alpha}(y)\\ &\quad - \Lambda_2^*(r+\alpha+\beta) \int_0^\infty du\, \eta_2 e^{-\eta_2 u}\, p_{r+\alpha+\beta} * p_{r+\alpha}(u+y)\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty d\gamma\, e^{-\beta\gamma} A_3(\alpha,\gamma,y) &= \int_0^\infty dx\, \eta_1 e^{-\eta_1 x}\, p_{r+\alpha+\beta} * p_{r+\alpha}(y-x)\\ &\quad - \widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1)\, p_{r+\alpha+\beta} * p_{r+\alpha}(y)\\ &\quad - \widehat{\Gamma}_2^*(r+\alpha+\beta,\eta_1) \int_0^\infty du\, \eta_2 e^{-\eta_2 u}\, p_{r+\alpha+\beta} * p_{r+\alpha}(u+y).\end{aligned}$$

Since $p_{r+\alpha+\beta} * p_{r+\alpha}(y)$ is a mixture of exponential distributions, we can evaluate the corresponding integrals explicitly. At first we find expressions for

$$\int_{-\infty}^\infty dy\, (e^y - k)^+ p_{r+\alpha+\beta} * p_{r+\alpha}(y).$$

This reduces to the integrals

$$\int_0^\infty dy\, e^{-\beta y}(e^y - k)^+ = \frac{k + (1-k)^+\beta}{\beta(\beta-1)} e^{-\beta(\log(k))^+},$$

and

$$\int_{-\infty}^0 dy\, e^{\beta y}(e^y - k)^+ = \left(\frac{1-k^{\beta+1}}{\beta+1} - \frac{k-k^{\beta+1}}{\beta}\right) 1_{\{k<1\}}.$$

To evaluate

$$\int_{-\infty}^\infty dy\, (e^y - k)^+ (p_{r+\alpha+\beta} * p_{r+\alpha})'(y),$$

we have to evaluate the same integrals.
Next we consider

$$\int_{-\infty}^{\infty} dy\, (e^y - k)^+ p_{r+\alpha+\beta} * p_{r+\alpha}(y-x).$$

This leads to the integrals

$$\int_{x}^{\infty} dy\, e^{-\beta(y-x)}(e^y-k)^+ = e^{\beta x}\left(\frac{1}{\beta-1}e^{-(\beta-1)(\log(k)\vee x)} - \frac{k}{\beta}e^{-\beta(\log(k)\vee x)}\right)$$

and

$$\int_{-\infty}^{x} dy\, e^{\beta(y-x)}(e^y-k)^+ = e^{-\beta x}\left(\frac{e^{(\beta+1)x}-k^{(\beta+1)}}{\beta+1} - \frac{e^{\beta x}k-k^{\beta+1}}{\beta}\right)1_{\{\log(k)<x\}}.$$

Further we have to evaluate

$$\int_{-\infty}^{\infty} dy \int_{0}^{\infty} du\, \eta_2 e^{-\eta_2 u} p_{r+\alpha+\beta} * p_{r+\alpha}(u+y)(e^y-k)^+,$$

which, with $(x)^- = \max(0,-x)$, leads to the integrals

$$\begin{aligned}\int_{-\infty}^{\infty} dy \int_{(-y)^+}^{\infty} & du\, \eta_2 e^{-\eta_2 u} e^{-\beta(u+y)}(e^y-k)^+ \\ &= \frac{\eta_2(1-e^{-(\eta_2+1)(\log(k))^-})}{(\beta-1)(\eta_2+1)} - \frac{k(1-e^{-\eta_2(\log(k))^-})}{\beta} + \frac{\eta_2 e^{-(\eta_2+\beta)(\log(k))^-}}{\eta_2+\beta}\frac{k^{1-\beta}}{(\beta-1)\beta}\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} dy \int_{0}^{(-y)^+} & du\, \eta_2 e^{-\eta_2 u} e^{\beta(u+y)}(e^y-k)^+ \\ &= I_{\{k<1\}}\left(\frac{\eta_2(1-k^{\eta_2+1})}{(\beta+1)(\eta_2+1)} - \frac{\eta_2(k^{\beta+1}-k^{\eta_2+1})}{(\beta+1)(\eta_2-\beta)} - \frac{k(1-k^{\eta_2})}{\beta} + \frac{k\eta_2(k^{\beta}-k^{\eta_2})}{\beta(\eta_2-\beta)}\right).\end{aligned}$$

At last consider

$$\int_{-\infty}^{\infty} dy \int_{0}^{\infty} dx\, \eta_1 e^{-\eta_1 x} p_{r+\alpha+\beta} * p_{r+\alpha}(y-x)(e^y-k)^+,$$

which leads to the integrals

$$\begin{aligned}\int_{0}^{\infty} dy \int_{0}^{y} dx\, \eta_1 e^{-\eta_1 x} e^{-\beta(y-x)}(e^y-k)^+ &= \eta_1\frac{1-e^{-(\eta_1-\beta)(\log(k))^+}}{\eta_1-\beta}\frac{k^{1-\beta}}{\beta(\beta-1)} \\ &\quad + \frac{\eta_1 e^{-(\eta_1-1)(\log(k))^+}}{(\beta-1)(\eta_1-1)} - \frac{ke^{-\eta_1(\log(k))^+}}{\beta}\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} dy \int_{y^+}^{\infty} dx\, \eta_1 e^{-\eta_1 x} e^{\beta(y-x)}(e^y-k)^+ &= \frac{\eta_1 e^{-(\eta_1-1)(\log(k))^+}}{(\beta+1)(\eta_1-1)} - \frac{ke^{-\eta_1(\log(k))^+}}{\beta} \\ &\quad + \frac{\eta_1 e^{-(\eta_1+\beta)(\log(k))^+ + (\beta+1)\log(k)}}{\beta(\beta+1)(\eta_1+\beta)}.\end{aligned}$$

7. A particular case: Geometric Brownian motion

In this section we show that for $\lambda = 0$ the Laplace transforms w.r.t. γ can be inverted explicitly and indeed reduce to the expressions for $L_{\alpha,\gamma}(x, y)$ obtained by Chesney et al. [10, Sec.5]. Indeed, since then $G(\theta) = \sigma^2/2\theta^2 + \mu\theta$ the roots are

$$\beta_1^\alpha = -\frac{\sqrt{\mu^2 + 2\sigma^2\alpha} + \mu}{\sigma^2} \quad \text{and} \quad \beta_2^\alpha = \frac{\sqrt{\mu^2 + 2\sigma^2\alpha} - \mu}{\sigma^2}.$$

We get

$$\begin{aligned}
\Gamma_1(\alpha, b) &= e^{-\beta_2^\alpha b} = e^{-\frac{\sqrt{\mu^2+2\sigma^2\alpha}-\mu}{\sigma^2}b},\\
\widehat{\Gamma}_1(\alpha, s) &= \frac{s\sigma^2}{s\sigma^2 - \mu + \sqrt{\mu^2 + 2\sigma^2\alpha}},\\
\Lambda_1(\alpha) = \beta_2^\alpha &= \frac{\sqrt{\mu^2 + 2\sigma^2\alpha} - \mu}{\sigma^2}
\end{aligned}$$

and

$$\Gamma_2(\alpha, b) = \Lambda_2(\alpha) = 0.$$

For $x < 0$ we get, similarly to equation (5),

$$L_{\alpha,\gamma}(x, y) = \Gamma_1(r + \alpha, -x)L_{\alpha,\gamma}(0, y),$$

and for $x > 0$, similarly to equation (6),

$$L_{\alpha,\gamma}(x, y) = A_1(\alpha, \gamma, x, y) - \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma\right] L_{\alpha,\gamma}(0, y).$$

For the derivative at $x = 0$ we get

$$\Lambda_1(r + \alpha)L_{\alpha,\gamma}(0, y) = A_2(\alpha, \gamma, y) - \lim_{\varepsilon\to 0} \frac{1 - \mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma\right]}{\varepsilon} L_{\alpha,\gamma}(0, y).$$

In this case the Laplace transforms of the involved terms are given by

$$\int_0^\infty d\gamma\, e^{-\beta\gamma} A_1(\alpha, \gamma, x, y) = p_{r+\alpha+\beta} * p_{r+\alpha}(y - x) - \Gamma_1^*(r + \alpha + \beta, x)p_{r+\alpha+\beta} * p_{r+\alpha}(y),$$

$$\int_0^\infty d\gamma\, e^{-\beta\gamma} A_2(\alpha, \gamma, y) = -(p_{r+\alpha+\beta} * p_{r+\alpha})'(y) + \Lambda_1^*(r + \alpha + \beta)p_{r+\alpha+\beta} * p_{r+\alpha}(y)$$

and

$$\beta \int_0^\infty d\gamma\, e^{-\beta\gamma}\, \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma\right] = \Gamma_1^*(r + \alpha + \beta, x),$$

$$\beta \int_0^\infty d\gamma\, e^{-\beta\gamma} \lim_{\varepsilon\to 0} \frac{1 - \mathbb{E}^\varepsilon\left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma\right]}{\varepsilon} = \Lambda_1^*(r + \alpha + \beta, x),$$

where

$$\begin{aligned} p_{r+\alpha+\beta} * p_{r+\alpha}(x) = {} & \left(\frac{2}{\sigma^2\beta\left(\beta_2^{r+\alpha} - \beta_1^{r+\alpha}\right)} e^{-\beta_2^{r+\alpha}x} - \frac{2}{\sigma^2\beta\left(\beta_2^{r+\alpha+\beta} - \beta_1^{r+\alpha+\beta}\right)} e^{-\beta_2^{r+\alpha+\beta}x} \right) 1_{\{x>0\}} \\ & + \left(\frac{2}{\sigma^2\beta\left(\beta_2^{r+\alpha} - \beta_1^{r+\alpha}\right)} e^{-\beta_1^{r+\alpha}x} - \frac{2}{\sigma^2\beta\left(\beta_2^{r+\alpha+\beta} - \beta_1^{r+\alpha+\beta}\right)} e^{-\beta_1^{r+\alpha+\beta}x} \right) 1_{\{x<0\}}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^\infty d\gamma\, e^{-\beta\gamma} A_2(\alpha,\gamma,y) = {} & \Bigg(\frac{\Lambda_1(r+\alpha) + \Lambda_1^*(r+\alpha+\beta)}{\beta\sqrt{\mu^2+2\sigma^2(r+\alpha)}} \exp\left(\frac{\mu - \sqrt{\mu^2+2\sigma^2(r+\alpha)}}{\sigma^2} y \right) \\ & - \frac{2}{\beta\sigma^2} \exp\left(\frac{\mu - \sqrt{\mu^2+2\sigma^2(r+\alpha+\beta)}}{\sigma^2} y \right) \Bigg) 1_{\{y>0\}} \\ & + \frac{\beta_1^{r+\alpha} + \Lambda_1^*(r+\alpha+\beta)}{\beta\sqrt{\mu^2+2\sigma^2(r+\alpha)}} \exp\left(\frac{\mu + \sqrt{\mu^2+2\sigma^2(r+\alpha)}}{\sigma^2} x \right) 1_{\{y<0\}}. \end{aligned}$$

With some effort, one can then show that the inverse Laplace transforms of the above expressions indeed coincides with the terms given in Chesney et al. [10, p.174-175][1].

8. A particular case: Compound Poisson model with upward exponential jumps

In this section we assume that $\mu < 0$, $\sigma = 0$, $p = 1$ and that $-\mu > \lambda/\eta_1$, which means that we have a drift to $-\infty$. This corresponds to the process considered by other techniques in Dassios and Wu [13], so that we can compare the two methods (note that [13] deals with a down-and-in option in a model with downward jumps, which corresponds to our up-and-in option in the model with upward jumps). In [13], at first an expression (containing an integral over a modified Bessel function) for $\mathbb{E}[e^{-\beta X_{e_\alpha}} 1_{\{\tau_\gamma < e_\alpha\}} | X_0 = x, x < 0]$ is provided. If $h(y,x,\gamma)$ denotes the inverse of this Laplace transform, then the Laplace transform (with respect to maturity time T) of the option price is given by

$$\int_{-\infty}^{-\log(k)} dy\, (e^{-y} - k)^+ h(y,x,\gamma).$$

So essentially [13] proposes to first take the double Laplace transform w.r.t. T and y, then numerically invert it and finally integrate the result w.r.t. y to obtain the option price.

On the other hand, our approach (applied to this model) avoids the integral over the modified Bessel for the double-Laplace transform; furthermore the needed integration w.r.t. y can be done explicitly before numerically inverting the double Laplace transform w.r.t. γ and T.

For the compound Poisson model with upward exponential jumps the Laplace transforms w.r.t. γ can not be inverted explicitly as in the pure diffusion case of Section 7, but nevertheless the formulas simplify significantly. In particular we get that

$$G(\theta) = \theta\mu + \lambda\left(\frac{\eta_1}{\eta_1 - \theta} - 1 \right)$$

---

[1] Note that there is a typographical error in [10], namely by the definition of $K_{\lambda,D}(a)$ on page 174, the resulting formula has to be multiplied by D; further a has to be replaced by $-a$.

which leads to the two roots

$$\beta_1^\alpha = \frac{\alpha+\lambda+\mu\eta_1+\sqrt{(\alpha+\lambda-\mu\eta_1)^2+4\mu\eta_1\lambda}}{2\mu}$$

$$\beta_2^\alpha = \frac{\alpha+\lambda+\mu\eta_1-\sqrt{(\alpha+\lambda-\mu\eta_1)^2+4\mu\eta_1\lambda}}{2\mu}$$

With $p \to 1$ and $\sigma \to 0$ we get (cf. [17] for a direct derivation)

$$\Gamma_1^*(\alpha,b) = e^{\beta_1^\alpha b} = \exp\Big\{\frac{\alpha+\lambda+\mu\eta_1+\sqrt{(\alpha+\lambda-\mu\eta_1)^2+4\mu\eta_1\lambda}}{2\mu}b\Big\},$$

$$\widehat{\Gamma}_1^*(\alpha,s) = \frac{s}{s-\beta_1^\alpha} = \frac{2\mu s}{2\mu s-\Big(\alpha+\lambda+\mu\eta_1+\sqrt{(\alpha+\lambda-\mu\eta_1)^2+4\mu\eta_1\lambda}\Big)},$$

and

$$\Gamma_2(\alpha,b) = \left(1-\frac{\beta_2^\alpha}{\eta_1}\right)e^{-\beta_2^\alpha b},$$

$$\widehat{\Gamma}_2(\alpha,s) = \left(1-\frac{\beta_2^\alpha}{\eta_1}\right)\frac{s}{s+\beta_2^\alpha}.$$

For $x \le 0$ we get, similarly to equation (5),

$$L_{\alpha,\gamma}(x,y) = \Gamma_2(r+\alpha,-x)I_{\alpha,\gamma}(y),$$

and for $x > 0$, similarly to (6),

$$L_{\alpha,\gamma}(x,y) = A_1(\alpha,\gamma,x,y) + \mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^- < \gamma\right]L_{\alpha,\gamma}(0,y).$$

Equation (7) reads

$$I_{\alpha,\gamma}(y) = A_3(\alpha,\gamma,y) + \int_0^\infty dx\,\eta_1 e^{-\eta_1 x}\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^- < \gamma\right]L_{\alpha,\gamma}(0,y).$$

It follows that

$$L_{\alpha,\gamma}(0,y) = \frac{A_3(\alpha,\gamma,y)}{\frac{1}{\Gamma_2(r+\alpha,0)} - \int_0^\infty dx\,\eta_1 e^{-\eta_1 x}\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^- < \gamma\right]}.$$

For the involved terms, we get the Laplace transforms

$$\int_0^\infty d\gamma\, e^{-\beta\gamma}A_1(\alpha,\gamma,x,y) = p_{r+\alpha+\beta} * p_{r+\alpha}(y-x) - \Gamma_1^*(r+\alpha+\beta,x)p_{r+\alpha+\beta} * p_{r+\alpha}(y),$$

$$\int_0^\infty d\gamma\, e^{-\beta\gamma}A_3(\alpha,\gamma,y) = \int_0^\infty dx\eta_1 e^{-\eta_1 x}p_{r+\alpha+\beta} * p_{r+\alpha}(y-x) - \widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1)p_{r+\alpha+\beta} * p_{r+\alpha}(y),$$

$$\beta\int_0^\infty d\gamma\, e^{-\beta\gamma}\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^- < \gamma\right] = \Gamma_1^*(r+\alpha+\beta,x),$$

$$\beta\int_0^\infty d\gamma\int_0^\infty dx\,\eta_1 e^{-\eta_1 x}e^{-\beta\gamma}\,\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^- < \gamma\right] = \widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1),$$

where

$$p_{r+\alpha+\beta} * p_{r+\alpha}(x) = \left(\frac{\eta_1 - \beta_2^{\alpha+r}}{-\mu\beta(\beta_2^{r+\alpha} - \beta_1^{r+\alpha})} e^{-\beta_2^{r+\alpha} x} - \frac{\eta_1 - \beta_2^{\alpha+r+\beta}}{-\mu\beta(\beta_2^{r+\alpha+\beta} - \beta_1^{r+\alpha+\beta})} e^{-\beta_2^{r+\alpha+\beta} x} \right) 1_{\{x>0\}}$$
$$+ \left(\frac{\eta_1 - \beta_1^{\alpha+r}}{-\mu\beta(\beta_2^{r+\alpha} - \beta_1^{r+\alpha})} e^{-\beta_1^{r+\alpha} x} - \frac{\eta_1 - \beta_1^{\alpha+r+\beta}}{-\mu\beta(\beta_2^{r+\alpha+\beta} - \beta_1^{r+\alpha+\beta})} e^{-\beta_1^{r+\alpha+\beta} x} \right) 1_{\{x<0\}}.$$

Similarly to [13] we get that the inverse Laplace transform of

$$\frac{1}{\beta}\widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1) = -\frac{2\mu\eta_1}{\beta\left(r+\alpha+\beta+\lambda-\mu\eta_1+\sqrt{(r+\alpha+\beta+\lambda-\mu\eta_1)^2+4\mu\eta_1\lambda}\right)}$$

is given by

$$-\int_0^\gamma dx\, \frac{\eta_1\mu}{x\sqrt{-\eta_1\lambda\mu}} e^{-x(r+\alpha+\lambda-\mu\eta_1)} I_1\left(2x\sqrt{-\eta_1\lambda\mu}\right),$$

where $I_1(x)$ denotes the modified Bessel function of the first kind ([2, p. 374]). Note that this corresponds to $\widetilde{P}(r+\alpha)$ in [13, p.6].

We will now show that our approach is consistent with the one of Dassios and Wu [13]. For this purpose, we first take the Laplace form w.r.t. y and then invert the result w.r.t. γ to obtain the corresponding expression of [13]. Concretely, let us hence look for the inverse Laplace transform w.r.t. β of the double Laplace transform

$$\int_{-\infty}^\infty dy\, e^{-sy} \int_0^\infty d\gamma\, e^{-\beta\gamma} A_3(\alpha,\gamma,y), \qquad -\beta_2^\alpha < s < -\beta_1^\alpha.$$

We start with the inverse of $\int_0^\infty dx\, \eta_1 e^{-\eta_1 x} p_{r+\alpha+\beta} * p_{r+\alpha}(y-x)$. Note first that

$$\int_{-\infty}^\infty dy\, e^{-sy} \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \frac{\eta_1 - \beta_2}{-\mu\beta(\beta_2-\beta_1)} e^{-\beta_2(y-x)} 1_{\{y-x>0\}}$$
$$+ \int_{-\infty}^\infty dy\, e^{-sy} \int_0^\infty dx\, \eta_1 e^{-\eta_1 x} \frac{\eta_1 - \beta_1}{-\mu\beta(\beta_2-\beta_1)} e^{-\beta_1(y-x)} 1_{\{y-x<0\}} = \frac{\eta_1}{\mu\beta(s+\beta_1)(s+\beta_2)}.$$

Hence we have to find the inverse Laplace transform of

$$\frac{\eta_1}{\mu\beta\left(s+\beta_1^{r+\alpha+\beta}\right)\left(s+\beta_2^{r+\alpha+\beta}\right)} = \frac{\eta_1}{\beta\left(\mu s^2 + s(\alpha+r+\beta+\lambda+\mu\eta_1) + (\alpha+r+\beta)\eta_1\right)},$$

which is given by

$$\frac{\eta_1}{\mu s^2 + s(\alpha+r+\lambda+\mu\eta_1) + (\alpha+r)\eta_1} \times \left(1 - \exp\left(-\frac{\gamma(s\alpha+\alpha\eta_1+r(s+\eta_1)+s\lambda+s(s+\eta_1)\mu)}{s+\eta_1}\right)\right).$$

Further we have to find the inverse of $\widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1)p_{r+\alpha+\beta} * p_{r+\alpha}(y)$. Note that

$$\int_{-\infty}^\infty dy\, e^{-sy} \left(\frac{\eta_1-\beta_2}{-\mu\beta(\beta_2-\beta_1)} e^{-\beta_2 y} 1_{\{y>0\}} + \frac{\eta_1-\beta_1}{-\mu\beta(\beta_2-\beta_1)} e^{-\beta_1 y} 1_{\{y<0\}}\right) = \frac{s+\eta_1}{\mu\beta(s+\beta_1)(s+\beta_2)}.$$

Hence we have to find the inverse Laplace transform of

$$\widehat{\Gamma}_1^*(r+\alpha+\beta,\eta_1)\frac{s+\eta_1}{\beta\left(\mu s^2+s(\alpha+r+\beta+\lambda+\mu\eta_1)+(\alpha+r+\beta)\eta_1\right)},$$

which is given by

$$\begin{aligned}
-\frac{\eta_1+s}{\mu s^2+s(\alpha+r+\lambda+\mu\eta_1)+(\alpha+r)\eta_1}&\left(\int_0^\gamma dx\,\frac{\eta_1\mu}{x\sqrt{-\eta_1\lambda\mu}}e^{-x(r+\alpha+\lambda-\mu\eta_1)}I_1\left(2x\sqrt{-\eta_1\lambda\mu}\right)\right.\\
&-\exp\left(-\frac{\gamma(s\alpha+\alpha\eta_1+r(s+\eta_1)+s\lambda+s(s+\eta_1)\mu)}{s+\eta_1}\right)\\
&\left.\times\int_0^\gamma dx\,\frac{\eta_1\mu}{x\sqrt{-\eta_1\lambda\mu}}e^{-x\left(\lambda\frac{\eta_1}{\eta_1+s}-(\eta_1+s)\mu\right)}I_1\left(2x\sqrt{-\eta_1\lambda\mu}\right)\right).
\end{aligned}$$

This eventually gives the Laplace transform of A_3 with respect to y:

$$\begin{aligned}
\int_{-\infty}^{\infty}dy\,e^{-sy}A_3(\alpha,\gamma,y)=&\frac{(\eta_1+s)\exp\left(-\frac{\gamma(s\alpha+\alpha\eta_1+r(s+\eta_1)+s\lambda+s(s+\eta_1)\mu)}{s+\eta_1}\right)}{\mu s^2+s(\alpha+r+\lambda+\mu\eta_1)+(\alpha+r)\eta_1}\\
&\times\left(\frac{\eta_1}{\eta_1+s}-\int_0^\gamma dx\,\frac{-\eta_1\mu}{x\sqrt{-\eta_1\lambda\mu}}e^{-x\left(\lambda\frac{\eta_1}{\eta_1+s}-(\eta_1+s)\mu\right)}I_1\left(2x\sqrt{-\eta_1\lambda\mu}\right)\right).
\end{aligned}$$

Note that

$$\frac{1}{\alpha}\mathbb{E}^x[e^{-sX_{e_\alpha}};\tau_\gamma<e_\alpha]=\frac{\Gamma_2(\alpha,-x)\ \int_{-\infty}^{\infty}dy\,e^{-sy}A_3(\alpha,\gamma,y)}{1-\Gamma_2(\alpha,0)\int_0^\infty dx\,\eta_1e^{-\eta_1x}\mathbb{E}^x\left[e^{-(r+\alpha)\nu_0^-};\nu_0^-<\gamma\right]}.$$

Dassios and Wu [13] express this formula in a slightly different way, but the two expressions can be reconciled: In [13] one has for $x<0$ (in our notation)

$$\begin{aligned}
\frac{1}{\alpha}\mathbb{E}^x[e^{-sX_{e_\alpha}};\tau_\gamma<e_\alpha]=&\frac{e^{-sx}}{\alpha+\lambda s/(\eta_1+s)+\mu s}\mathbb{E}\left[\exp\left(-\left(\alpha-\lambda\left(\frac{\eta_1}{\eta_1+s}-1\right)+\mu s\right)\tau_\gamma^*\right)\right]\\
=&\frac{1}{\alpha+\lambda s/(\eta_1+s)+\mu s}\frac{\eta_1+s+\nu_{\beta^*}^-}{\eta_1+s}e^{-\beta^*\gamma-x\nu_{\beta^*}^--sx}\frac{\overline{P}_{21}^*(\gamma)}{1-\hat{P}_{12}^*(\beta^*)\tilde{P}_{21}^*(\beta^*)},
\end{aligned}$$

where τ_γ^* is the stopping time for a process with $c^*=c$, $\eta_1^*=\eta_1+s$ and $\lambda^*=\lambda\eta_1/(\eta_1+s)$ and

$$\begin{aligned}
\nu_{\beta^*}^-=&-\frac{\sqrt{\left(-\mu(\eta_1+s)+\left(\alpha-\lambda\left(\frac{\eta_1}{\eta_1+s}-1\right)+\mu s\right)+\lambda\frac{\eta_1}{\eta_1+s}\right)^2+4\mu\lambda\eta_1}}{-2\mu}\\
&-\frac{\left(-\mu(\eta_1+s)-\left(\alpha-\lambda\left(\frac{\eta_1}{\eta_1+s}-1\right)+\mu s\right)-\lambda\frac{\eta_1}{\eta_1+s}\right)}{-2\mu}\\
=&\frac{\sqrt{(\alpha+\lambda-\mu\eta_1)^2+4\mu\lambda\eta_1}-\alpha-\lambda-\mu\eta_1}{2\mu}-s=-\beta_2^\alpha-s
\end{aligned}$$

with

$$\hat{P}_{12}^*(\beta^*)=\frac{\eta_1+s+\nu_\beta^-}{\eta_1+s}=\left(1-\frac{\beta_2^\alpha}{\eta_1}\right)\frac{\eta_1}{\eta_1+s}=\Gamma_2(\alpha,0)\frac{\eta_1}{\eta_1+s}$$

and

$$\tilde{P}^*_{21}(\beta^*) = \frac{\eta_1 + s}{\eta_1} \int_0^\gamma dt \, \frac{-\mu\eta_1}{\sqrt{-\mu\lambda\eta_1}} e^{-(\alpha+\lambda-\mu\eta_1)t} t^{-1} I_1\left(2t\sqrt{-\mu\lambda\eta_1}\right).$$

It follows that

$$\hat{P}^*_{12}(\beta^*)\tilde{P}^*_{21}(\beta^*) = \Gamma_2(r+\alpha, 0) \int_0^\infty dx \, \eta_1 e^{-\eta_1 x} \mathbb{E}^x \left[e^{-(r+\alpha)\nu_0^-}; \nu_0^- < \gamma\right].$$

In [13] it is shown that

$$\overline{P}^*_{21}(\gamma) = 1 - \frac{\eta_1 + s}{\eta_1} \int_0^\gamma dt \, \frac{-\mu\eta_1}{\sqrt{-\mu\lambda\eta_1}} e^{-\left(\lambda\frac{\eta_1}{\eta_1+s} - \mu(\eta_1+s)\right)t} t^{-1} I_1\left(2t\sqrt{-\mu\lambda\eta_1}\right)$$

and

$$\begin{aligned} \frac{\eta_1 + s + \nu^-_{\beta^*}}{\eta_1 + s} e^{-\beta^*\gamma - x\nu^-_{\beta^*} - sx} &= \frac{\eta_1 - \beta_2^\alpha}{\eta_1 + s} e^{x\beta_2^\alpha} \exp\left(-\gamma\left(\alpha - \lambda\left(\frac{\eta_1}{\eta_1+s} - 1\right) + \mu s\right)\right) \\ &= \Gamma_2(\alpha, -x)\frac{\eta_1}{\eta_1+s} \exp\left(-\gamma\frac{(\eta_1+s)\alpha + \lambda s + \mu s(\eta_1+s)}{\eta_1+s}\right). \end{aligned}$$

So it just remains to show that

$$\frac{1}{\alpha + \lambda s/(\eta_1+s) + \mu s} = \frac{(\eta_1+s)}{\mu s^2 + s(\alpha+\lambda+\mu\eta_1) + \alpha\eta_1},$$

but the latter follows from simple algebraic computations.

9. Numerical example

We now provide a numerical illustration to show the effectiveness of the resulting procedure to obtain the price of the Parisian option. Concretely, we collect the terms in formula (15) for the Laplace transform of the option price w.r.t. T obtained in the paper, which are given either directly or in terms of their Laplace transforms w.r.t. γ. We first numerically invert these latter Laplace transforms and use the results to finally perform the numerical Laplace transform w.r.t. T of (15).

Assume the barrier $L = 90$ and parameters $k = 95$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$, $\lambda = 4$, $\eta_1 = \eta_2 = 10$, $p = 17/40$ and $r = 493/9900 \approx 0.049798$. As Bernard et al. [5] who deal with the diffusion case, we use different values of γ and starting capital S_0. Figure 1 depicts some paths of the resulting asset process with $S_0 = 80$. In Figure 2 we provide a plot of the option price for different values of γ and S_0. We can see that the option price is monotone decreasing in γ and monotone increasing in S_0. In Table 1 we give option prices for different values of γ and S_0, where we used the GWR method of Abate and Valko (cf. [1]) for the calculation of the inverse Laplace transform, which consists of the Gaver-Stehfest algorithm together with a special error extrapolation. The algorithm is a numerical approximation of the Post-Widder inversion formula for Laplace transforms and hence has the advantage that we only need to know the Laplace transform for real-valued arguments. A disadvantage of this method is that we have to use multi-precision calculations to avoid numerical instability. Since we have to use two successive Laplace inversions, the inner inversion has to provide very accurate results, as it is used as an input for the outer inversion. The GWR algorithm uses two parameters: a parameter M which controls the number of points at which the Laplace transform is evaluated and a parameter *precin* which specifies the working precision (in digits). The following parameters

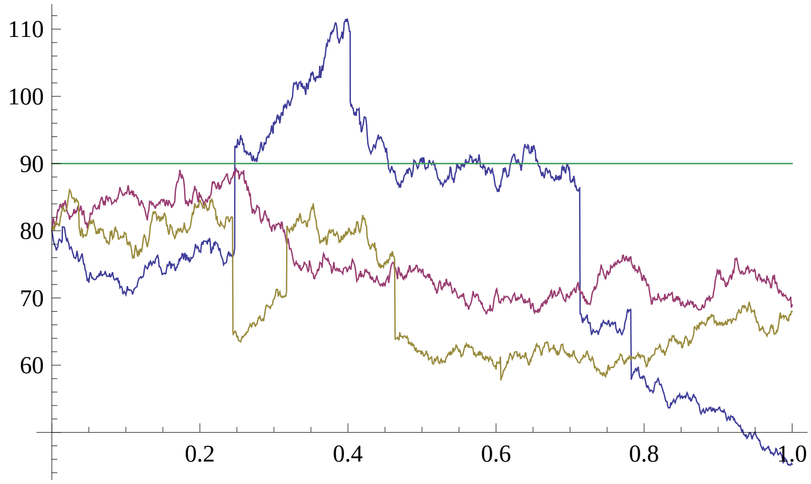

FIGURE 1. Some sample paths

work quite well for the considered problem: $M_i = 41$ and precin $= 94.5$ for the inner inversion and $M_o = 15$ and precin $= 33$ for the outer inversion (a Mathematica implementation of this algorithm can e.g. be found at `http://library.wolfram.com/infocenter/MathSource/4738/`). In Table 2 we give the evaluation time of each price.

One should note that the evaluation time is significantly shorter if S_0 is below the barrier (due to the simpler formula). Furthermore, the evaluation time also decreases with increasing γ, which is due to the fact that a major part of the evaluation time is used for the determination of the roots of $G(\theta) = r + \alpha + \beta$. As for the GWR method one has to evaluate the roots at $r + \log(2)(i + j/\gamma)$ (where $i = 1, \dots, 2M_o$ and $j = 0, \dots, 2M_i$), this means that for, say, $\gamma \in \{1/360, 1/52\}$ one has to evaluate 2490 different roots, whereas for $\gamma = 3/12$ one only has to evaluate 358 different roots.

To estimate the accuracy of our method, we evaluated the option prices with increased parameters for the GWR method. The resulting relative errors are provided in Table 3. We see that with increasing γ the accuracy of the method is decreasing.

For comparison, we also give simulation results for the Parisian option price under these model assumptions. We employ 10 million sample paths and a discretization of $1/1000$ for the diffusion (resulting in a simulation time of about 1500 sec for each value). As can be seen from Table 4, the achieved accuracy can then still not compete with the one of the numerical method proposed in this paper, although simulation takes at least 75 times as much computation time (for possible improvements of the simulation method based on hitting times, see e.g. Anderluh and van der Weide [3]). Note that the simulation was performed in C++, whereas the numerical procedure using Laplace inversion was implemented in Mathematica, so that the performance potential of the numerical method is even higher.

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$S_0\backslash\gamma$	1/360	1/52	2/52	1/12	3/12
80	6.98	6.97	6.95	6.88	6.27
82	7.84	7.83	7.81	7.74	7.14
84	8.76	8.75	8.74	8.67	8.08
86	9.74	9.73	9.72	9.66	9.09
88	10.77	10.77	10.76	10.71	10.17
90	11.87	11.86	11.86	11.81	11.32
92	13.02	13.01	13.01	12.97	12.52
94	14.22	14.22	14.21	14.19	13.79
96	15.47	15.47	15.47	15.45	15.1
98	16.77	16.77	16.77	16.76	16.46
100	18.12	18.12	18.12	18.11	17.86

Table 1. Table of prices for Parisian options for different values of γ and S_0

$S_0\backslash\gamma$	1/360	1/52	2/52	1/12	3/12
80	18.35 s	18.67 s	17.37 s	13.68 s	11.66 s
82	18.47 s	18.51 s	17.36 s	13.69 s	11.67 s
84	18.48 s	18.55 s	17.43 s	13.64 s	11.66 s
86	18.39 s	18.37 s	17.27 s	13.57 s	11.58 s
88	18.37 s	18.43 s	17.31 s	13.59 s	11.59 s
90	18.39 s	18.49 s	17.31 s	13.58 s	11.62 s
92	21.56 s	21.67 s	20.42 s	16.95 s	14.69 s
94	21.92 s	22.05 s	20.61 s	16.84 s	14.78 s
96	21.75 s	21.79 s	20.87 s	17.06 s	15.00 s
98	21.68 s	21.71 s	20.59 s	16.87 s	14.85 s
100	21.63 s	21.72 s	20.73 s	16.83 s	14.84 s

Table 2. The evaluation times for the values in Table 1 (with Mathematica 7.0)

$S_0\backslash\gamma$	1/360	1/52	2/52	1/12	3/12
80	2.0×10^{-11}	2.5×10^{-11}	8.1×10^{-11}	3.2×10^{-8}	3.1×10^{-8}
82	1.1×10^{-10}	1.5×10^{-11}	8.3×10^{-11}	8.3×10^{-8}	3.1×10^{-6}
84	4.2×10^{-12}	1.0×10^{-11}	2.9×10^{-11}	1.6×10^{-9}	9.2×10^{-6}
86	4.1×10^{-13}	2.5×10^{-11}	9.7×10^{-11}	1.3×10^{-9}	1.5×10^{-5}
88	9.9×10^{-13}	1.1×10^{-11}	1.5×10^{-10}	8.3×10^{-9}	2.4×10^{-5}
90	7.5×10^{-13}	9.7×10^{-12}	1.2×10^{-10}	1.5×10^{-8}	6.7×10^{-5}
92	8.8×10^{-14}	3.0×10^{-12}	2.1×10^{-10}	1.4×10^{-8}	1.9×10^{-5}
94	5.6×10^{-15}	3.0×10^{-12}	1.4×10^{-10}	2.4×10^{-8}	2.0×10^{-5}
96	3.1×10^{-13}	1.6×10^{-12}	1.1×10^{-9}	3.5×10^{-8}	2.1×10^{-5}
98	6.1×10^{-14}	2.7×10^{-11}	6.3×10^{-9}	2.0×10^{-8}	3.3×10^{-5}
100	1.1×10^{-13}	3.9×10^{-11}	3.6×10^{-10}	8.7×10^{-9}	3.4×10^{-4}

Table 3. Relative errors of the option price with respect to an increased accuracy

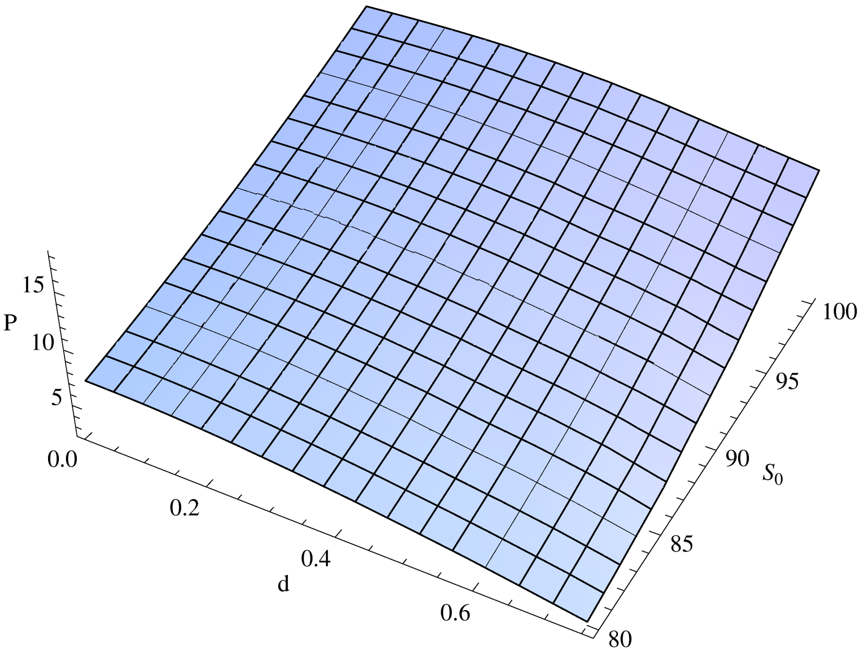

FIGURE 2. A plot of the option price with respect to up time and start price

$S_0\backslash\gamma$	1/360	1/52	2/52	1/12	3/12
80	6.5×10^{-4}	7.3×10^{-4}	1.8×10^{-4}	5.1×10^{-4}	4.6×10^{-3}
82	8.8×10^{-4}	3.1×10^{-4}	1.1×10^{-3}	2.3×10^{-4}	4.8×10^{-4}
84	1.2×10^{-4}	1.3×10^{-3}	6.6×10^{-5}	7.7×10^{-4}	3.2×10^{-3}
86	6.1×10^{-5}	3.5×10^{-4}	3.0×10^{-4}	9.0×10^{-4}	2.9×10^{-3}
88	4.0×10^{-4}	8.5×10^{-4}	2.1×10^{-3}	1.9×10^{-3}	3.6×10^{-3}
90	1.2×10^{-3}	3.7×10^{-4}	2.0×10^{-4}	1.9×10^{-3}	1.2×10^{-3}
92	7.1×10^{-4}	10.0×10^{-4}	8.0×10^{-4}	1.2×10^{-3}	1.3×10^{-3}
94	2.0×10^{-4}	1.3×10^{-3}	7.6×10^{-4}	1.2×10^{-3}	1.3×10^{-3}
96	5.8×10^{-4}	1.3×10^{-4}	5.0×10^{-5}	3.6×10^{-4}	1.5×10^{-3}
98	6.0×10^{-4}	2.9×10^{-4}	1.4×10^{-3}	8.3×10^{-4}	6.7×10^{-4}
100	2.7×10^{-4}	6.6×10^{-4}	1.2×10^{-4}	5.3×10^{-4}	4.1×10^{-4}

TABLE 4. Relative errors of the simulated option price with respect to the option price with Laplace transform inversion

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