

FINANCE RESEARCH SEMINAR SUPPORTED BY UNIGESTION

“Dynamic Leveraging-Deleveraging Games”

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Friday, November 24, 2017, 10:30 - 12:00
Room 126, Extranef building at the University of Lausanne

Dynamic Leveraging-Deleveraging Games ^{*}

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November 22, 2017

Abstract

We introduce a new mechanism for leverage dynamics, based on a multi-period game of lenders with differentiated beliefs about the firm's fundamental returns. The game features strategic substitutability for low existing leverage and one-sided strategic complementarity for high existing leverage. The resulting leverage process exhibits a mean-reverting regime around a long-run level, as long as it stays below an instability level. Above the instability level, leverage becomes explosive. We validate our model empirically using aggregate returns of financial firms over the 10 year period 2001 – 2010. Our model predicts the leveraging/deleveraging of this period, and in particular the 2008 collapse in short term debt.

Keywords: Systemic Risk, Debt Capacity, Funding Liquidity Risk, Nash Equilibrium, Heterogeneous Beliefs, Bayesian Update.

^{*}We thank Phil Dybvig, Pete Kyle (discussant), George Papanicolaou, Duane Seppi, Vikrant Vig, and the participants of the IPAM Culminating Workshop of the program Broad Perspective and New Directions in Financial Mathematics (2015), the Financial Stability Conference, Cleveland Fed and OFR (2015) and seminar participants at Columbia University (2015), UT Austin (2016), LSE (2016), Imperial College (2016), Carnegie Mellon University (2017), LBS (2017) for helpful comments. Andreea Minca is partly funded through NSF CAREER award #1653354.

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1 Introduction

The last financial crisis provides prominent examples of excessive bank and fund leveraging using short term debt, that ended with collapse, see e.g., Acharya et al. (2011), Brunnermeier (2009), Duffie (2010), Gorton and Metrick (2012), Financial Crisis Inquiry Commission (2011) and led to systemic crises. A less recent example is the LTCM demise, see e.g. Rubin et al. (1999). We provide a new mechanism of asset-based leverage dynamics. It is the lenders' aggregate decision that determines how much debt, and consequently leverage, can the firm attain based on its assets. Our goal is to endogenize the dynamics of the leverage and of the debt capacity of the asset. The same game of the lenders - the leveraging/deleveraging game— played period after period generates the entire leverage dynamics. We seek to characterize the level where lenders push the firm into a deleveraging spiral.

We build a model which is applicable to a variety of debt maturities, e.g., one day, one month, one quarter. What makes the debt short term is that the maturity of the debt contract matches the frequency with which the lenders can observe the firm's performance, and lenders can thus react to the firm's shocks.¹ As model input we have the fundamental trajectory of log-returns of the firm's investment strategy, which is fixed. Every potential lender has a belief about the mean log-return in the next period. The cross distribution of beliefs is common knowledge, but the real world drift is not known. Our baseline case is the case of dogmatic agents whose cross distribution of beliefs is constant. In the last part of the paper we introduce learning, and allow the belief distribution to change over time.

At any (discrete) time, potential lenders take synchronous decisions whether to finance or not the borrower's asset (refinance in the case of existing lenders). They have an option to finance and the value of this option varies with lenders' beliefs about the firm's risk.² A marginal lender (as function of the existing leverage) is one for whom the option to lend equates the outside option (which is riskless cash with zero return). The marginal lender changes in each period, and determines the debt capacity of the borrower's asset.

The firm expands (contracts) its asset using the net inflow (outflow) of debt and there may be transaction costs. We can think of a large leveraged hedge fund (or dealer bank) that pursues a core strategy. The heterogeneous lenders are the only agents in the model, and they decide the in and out-flows of debt. Given these flows, the management expands or reduces the asset position. Therefore, the applicability of our model is for the case when the constraints on leverage imposed by the lenders are stricter than the constraints that would be imposed the management. For our purposes, we can think of the firm as a highly optimistic agent, whose self-imposed constraint on leverage would always be less strict than the constraint imposed by its lenders. The example of LTCM where the management pursued their core strategy until the fall of the fund naturally comes to mind. The game continues until either maturity, when the firm is liquidated, or until default, which occurs if the entire asset cannot cover a net debt outflow. In case of default, the asset is liquidated and proportionally distributed to outstanding lenders.

Our game-theoretic framework originates in Krishenik et al. (2015), who study a rollover game for the debt provision to a sovereign borrower. Krishenik et al. (2015) provides an alternative to the global games setting of Carlsson and Van Damme (1993), Morris and Shin (2001). It is more amenable to data and more suitable for a dynamic model: heterogeneous beliefs about the future evolution of the fundamentals replace the noisy observations of the fundamentals. Here we keep the advantages of this framework, while we extend significantly the game. As a novel ingredient,

¹ Here the lenders do not have exogenous liquidity shocks.

² A potential lender's option equates in value the interest rate minus a put option valued as in Merton (1974).

we have the firm's asset. The debt capacity of one unit of the asset is determined in an equilibrium of the lender game.

Our leveraging-deleveraging game features complex dependencies that are novel in the literature. The value of lenders' options to finance (or refinance) the firm's asset change with existing leverage (see Figure 5). When the existing leverage is low, the payoffs of pessimistic lenders decrease when others lend as well. There are two sources for this strategic substitutability, totally absent in previous literature. First, under pessimists' belief, more debt increases default probability (because the firm would become more leveraged for the next period and they expect low asset returns). Second, in the case of default the firm's assets would be shared among more lenders, so the pessimists' expected recovery rates decrease. (The optimists are indifferent to the other lenders: under their belief, they hold a default-free contract).

In turn, when the existing leverage is high, then the firm cannot survive unless a large fraction of the current lenders roll over. Only in this regime the game behaves like a rollover game and features one-sided strategic complementarity similarly to Goldstein and Pauzner (2005), Krishenik et al. (2015): payoffs increase sharply with raised debt (up to a point where debt is sufficient for the firm to remain liquid). The alternation of leveraging and deleveraging phases and the roles played by pessimists and optimists are reminiscent of a leverage cycle in the collateral equilibrium model of Geanakoplos (2010), see also Fostel and Geanakoplos (2013) and the references therein for a review of this literature.

The main contribution of this paper is to endogenize the dynamics of the asset-based leverage of a large borrower (and the debt capacity of the asset) and to fully determine the regimes of this leverage process: we show that asset-based leverage is mean reverting around a long-run level, and explosive above an instability level. The intuition comes from the changing nature of the lenders' game from strategic substitutability to one sided strategic complementarity: When leverage is below the instability level, the firm is not in danger of default, and the strategic substitutability in the payoff structure acts as a counterbalance on leverage, which is pushed down to the long-run level. If leverage reaches above the instability level, then it becomes explosive: the one-sided strategic complementarity leads to spiraling effects that end in default. Default technically happens when leverage hits a ceiling. Determining the regimes of the leverage yields early warning indicators of default: when the leverage deviates from the long-run level and reaches above the instability level, then in expectation it will reach the debt ceiling because of the regime change from mean-reverting to explosive. We can moreover quantify sustainable debt levels: a wide mean-reverting regime around the long run level is tantamount to stable short term debt.

We generalize the dynamic debt run literature Carmona et al. (2017), He et al. (2017), He and Xiong (2012a), Krishenik et al. (2015), Liang et al. (2015, 2014), which is focused on deleveraging (debt runs). Our game has three main features: it is dynamic, lenders' decisions are synchronous and the lenders' game drives both leveraging and deleveraging. Equally important, we have endogenous recovery rates in default, and these drive the strategic complementarity/substitutability profile of the game. With the exception of Krishenik et al. (2015), Liang et al. (2014, 2015) who can allow for synchronous decisions in the global games setting, most other works use a staggered debt structure and thereby insulate their models from the multiplicity of equilibria. However, it is valuable to allow for synchronous decisions because short-term debt is the outcome of a maturity race, see Brunnermeier and Oehmke (2013). Likely, at the end of this race, many of the lenders will have a similar contract. It is anecdotal that in the recent crisis, large banks saw tens of billions of liquidity withdrawn in a matter of days, so there were large-scale synchronous decisions. Here we fully expose our model to equilibria multiplicity.

Since prior works are focused on deleveraging, they find a "debt run barrier" Liang et al. (2015) or the "run threshold" He and Xiong (2012a): if the fundamental process touches these barriers

or thresholds, then a debt run ensues. Touching the barrier is non-anticipative in these previous works. The “ceiling” in our model corresponds to these debt-run barriers. However, our analysis exhibits two other levels that are critical to the understanding of leverage stability: the long run level and the instability level. Starting from zero debt, leverage reaches its long-run level and there is a leveraging phase as the firm expands its asset position according to the provided debt. After this initial leveraging phase, leverage is stable for a while as it mean-reverts: lenders adjust leverage (as outcome of their game) in response to asset returns. However, above the instability level, it is no longer possible to mean-revert: existing leverage is too high, and a deleveraging spiral ensues with the expectation to end in default (leverage is explosive and expected to reach the debt ceiling).

Our analysis thus goes far beyond the default characterization in terms of the barrier (or ceiling in our case) as we fully characterize the regimes of leverage and we quantify the stability of the leverage. These do not have a correspondent in the literature. Default becomes anticipative in our model: leverage is expected to reach the ceiling as soon as it deviates from the long run-level and crosses the instability level: If leverage ratio tomorrow is an endogenous and, as we prove, convex³ function of the debt-to-asset ratio today, then its largest fixed point is the start of the spiral of lender withdrawals in which the leverage ratio switches from being mean-reverting to explosive. Reaching the instability level is thus an early indicator of default, and the collapse in debt capacity occurs over the time lag in which the debt-to-asset ratio process increases from the long run level level to the instability level (and further to the ceiling).

We provide a powerful model that can be calibrated to real data. We provide a “proof of concept” on how to use the model on real data as we use the financial commercial paper as a case study of short term debt. The input is the aggregate fundamental returns of the securities dealers in the US over a 10 year period. Of course, our model is intended for one large firm and not for a sector, but publicly available data is on the aggregate level. Over this period, the number of dealers is stationary, and we think of the aggregate as a “representative” dealer. Our model predicts the leveraging/deleveraging witnessed over this period, and in particular the financial commercial paper collapse of 2008, see Figure 1. We obtain similar results when using a constant spread equal to the average spread over the period 2001 – 2005. The difference in the model-predicted debt using the actual spreads and the constant spread is small during the 2007 – 2008 period, in which the actual spreads strongly increased. This implies that the (short) put option value for the marginal lender increases very fast after a sequence of negative returns (due to convexity effects). The implication is that increasing spreads during the deleveraging spiral has little effect on preventing the debt collapse.

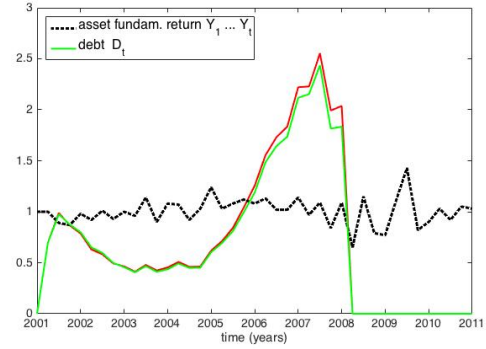
One assumption that leads to the elimination of multiple equilibria is that interest rates are constant.⁴ Spreads for short-term debt such as commercial paper did not fluctuate a lot before the collapse of the commercial paper outstanding notional, see e.g., Kacperczyk and Schnabl (2010) and Figure 8. The approximation by constant interest rates is thus valid in the mean-reverting regime of our model, when short term debt provision is stable. More importantly, in light of Figure 1 we do not expect that endogenizing interest rates would have any effect on the leverage regimes. Once a deleveraging spiral starts, payoffs of pessimistic lenders have little sensitivity to interest rates, and indeed during the crisis, spread increases did not prevent the debt collapse.

³Convexity holds under the assumption that the belief distribution has light right tails, which a natural assumption.

⁴Most literature features constant interest rates. Exceptions are Schroth et al. (2014) and the one period-model Jarrow et al. (2016). Given that it is a rollover game with asynchronous decisions, it is sufficient in Schroth et al. (2014) to set the interest rates such that maturing lenders have constant value one of their bond. In Jarrow et al. (2016), the authors study Stackelberg equilibria in a one period game where the firm chooses the optimal interest offered on debt and its cash holdings strategy and then lenders decide on the total debt. The main difficulty there is that that firm optimizes under uncertainty about the lenders’ equilibrium, and multiplicity of equilibria cannot be eliminated.



(a) Financial outstanding commercial paper (data not used)



(b) (Model) Debt dynamics with real-world spreads (red) vs. constant spread equal to the average spread over 2001 – 2005 (green). Real-world returns of FINRA members used as model input is plotted in dashed black line.

Figure 1: Real world leveraging/deleveraging vs. model predictions.

Our main technical contribution is to provide a refinement that leads to a unique equilibrium. We are building on the game-theoretic foundations established in Krishenik et al. (2015), where we show uniqueness of an equilibrium for a rollover game with synchronous decisions by lenders with heterogeneous beliefs. The proof technique there relies on eliminating weakly dominated strategies. Eliminating weakly dominating strategies is not enough to obtain uniqueness here, but we can use those results to eliminate the multiplicity of equilibria in which the firm defaults. There remain two cutoff equilibria in which the firm survives: the presence of default recovery rates leads to non-monotonous payoffs for a marginal lender with respect to the total debt. When the debt is high, the marginal lender is pessimistic and her payoff decreases with debt. When the debt is low, the marginal lender is optimistic and her payoff increases with debt. There are two possible marginal lenders that have the cutoff expected return zero. The technical innovation in this paper is the concept of Strongly ϵ -coalition proof equilibria, and the cutoff equilibrium with the lowest debt does not survive this refinement: if a small coalition of lenders just below the cutoff could coordinate to invest, then debt would increase and the entire coalition (along with the marginal lender) would see increasing payoffs.

Our paper is also related to the literature on optimal capital structure and endogenous bankruptcy, see e.g. Leland and Toft (1996). There, it is the firm who can choose its amount of debt. Here, it is lenders who collectively decide on the debt they provide and at the same time they determine the default condition. We assume that the firm is more optimistic than a large fraction of the lenders. The ensemble of lenders will impose a stricter constraint on leverage than the firm would set. (This constraint imposed by lenders comes in the form of the long-run leverage). The firm's dynamic optimization problem in the optimal capital structure literature is replaced here by the lenders' dynamic game. Our game can be thought of as a "game of timing", as investigated in Carmona et al. (2017). They are focused on convergence results for games with finite numbers of players in a game of timing with strategic complementarities. Here our setup is directly with a continuum of players and we are focused on refinements that lead to interpretable "barriers" when the game has both strategic complementarities and substitutabilities.

There is also an important recent literature on the optimal governance in presence of risk posed by the liability holders. For example, Cheng and Milbradt (2012) study the optimal debt

maturity. Like other works that focus on the lenders' debt provision game, here we restrict the firm from issuing equity. The firm's optimal capital structure (with exogenous debt capacity risk) is solved in He and Xiong (2012b). The one period model Jarrow et al. (2016) provides a motive for optimal cash holdings under heterogeneous lenders: cash allows the firm to increase its debt capacity and can be a more effective tool than raising the interest rate, in the sense that payoffs of pessimists are more sensitive to the cash holdings than to the interest rate. We conjecture that optimizing cash holdings would also affect the leverage dynamics more than interest rates, but these optimization problems of the firm deserve a full separate treatment. However, understanding the regimes of leverage induced by the lenders is a critical step and we expect that these regimes would perdure: the switch from strategic substitutability to complementarity in the payoffs of pessimistic lenders will remain a key driver.

The paper is organized as follows. Section 2 presents the model for the firm and introduces the lenders' game. Section 3 analyzes the games' equilibria and gives the uniqueness result. Section 4 contains the results on the regimes of the endogenous leverage process. Section 5 validates the model empirically by using as input the real world fundamental returns, aggregated over all FINRA members, and finally Section 6 illustrates the dynamic behavior of our model under a variety of parameters.

2 The model

We consider a firm that funds itself through short-term debt provided by a continuous lender base. The lender base is understood as potential lenders, and the size of the lender base tracks the firm's size. The actual lending results from a debt provision game. Time is discrete and there is a finite horizon T . The firm invests all available funding in a portfolio of risky assets with given risk. Lenders' beliefs about this risk differ.

2.1 Fundamentals

Fundamental trajectory. We start from a fundamental trajectory of the asset return, observable on a discrete time grid, $t = 1, 2, \dots, T$, when the lender base can observe the firm's asset performance *and* can make decisions to invest, withdraw or rollover.⁵ We denote by $\log Y_1, \dots, \log Y_T$ the sequence of fundamental log-returns of the firm's asset, and we assume that Y_t are independent log-normal random variables with $\text{Var}[\log Y_t] = \sigma^2$. Under the real-world probability measure \mathbb{P} the expected fundamental log-return is μ : $\mathbb{E}[Y_t] = e^\mu$.

We assume that there is a proportional cost both to liquidate and to purchase the asset, not necessarily the same. Liquidating the asset produces a loss equal to a fraction $\alpha \in [0, 1)$ of the traded volume if forced liquidation takes place before $t < T$. There are no liquidation costs at time T . The assumption of fixed, proportional, and known liquidation cost is common in the literature. We denote by $\alpha_1 \geq 0$ the corresponding cost when there are asset purchases (the asset is liquidated at time T , so the cost for asset purchases is relevant only for $t < T$). The firm's cash proceeds from liquidating an amount u of the asset value at time $t < T$ is given by

$$f_t(u) := \begin{cases} u(1 - \alpha \mathbf{1}_{t < T}) & \text{if } u \geq 0 \\ u(1 + \alpha_1) & \text{if } u < 0. \end{cases}$$

⁵The frequency thus corresponds to the minimum between the financing frequency and the performance reporting frequency.

Note that $f_t^{-1}(u) = \frac{u}{(1 - \mathbb{1}_{\{u \geq 0, t < T\}} \alpha + \mathbb{1}_{\{u < 0\}} \alpha_1)}$ and we have positive homogeneity: $f_t(\gamma u) = \gamma f_t(u)$ and $f_t^{-1}(\gamma u) = \gamma f_t^{-1}(u)$ for all $u \in \mathbb{R}$ and $\gamma \geq 0$.

Lenders' beliefs and maximum exposure to the firm. We assume that the lenders scale the maximum exposure to the firm linearly with the firm's size. We do not track the evolution of the lenders' wealth and in particular they consume any interest they may receive. Their decision is not driven by their own liquidity shocks but rather by their valuation of the option to finance the asset. Repo lending resembles most this setting: the lender only funds up to the value of the asset and the question in the case of a single lender is her "haircut" choice, namely the percentage of loan per asset value. With heterogenous lenders we will not consider the individual haircuts imposed by the lenders. Rather, we will have a notion of aggregate haircut, as some potential lenders will fully finance and other will not finance the asset at all. The fraction of lenders that finance the asset will give the haircut. If all lenders choose to finance the asset then there is no haircut at all: the raised debt is the same value as the firms' asset.

We start with a distribution Φ_0 on the set of beliefs $\mathcal{B} := \mathbb{R}$ and we draw randomly and independently of everything else the belief $b_0(a)$ of agent a from the distribution Φ_0 , known by all agents. At each time step t , we allow (but not oblige) lenders to update their beliefs, and let Φ_t the belief distribution at time t . Letting V_t the size of the firm's asset at time t , the financing that can be provided by agents with belief higher than b is

$$V_t \Phi_t(b). \quad (1)$$

The linear scaling assumption that will allow us to preserve the homogeneity of the model and describe the evolution of the firm in terms of a single state variable. It can be relaxed to more sophisticated dependencies at the expense of an increase in dimension of the state space. We prefer the linear dependence assumption because it implies homogeneity in firms' trajectories, which is reasonable given our setup with proportional transaction costs and no price impact. The resulting balance sheets scale linearly with the initial capital at any time: two firms with the same characteristics, i.e., asset returns and belief distribution among their lenders, will have the same trajectories up to a scaling factor.

An agent with *belief* b at time t measures risk using a probability measure \mathbb{P}^b under which the fundamental log-return $\log Y_t$ is independent of everything else, with $\text{Var}^b[\log Y_t] = \sigma^2$ and $\mathbb{E}^b[Y_t] = e^b$. There are two interpretations of the notion of beliefs.

Differentiated beliefs as differences of opinion. Under the first interpretation, agents are risk neutral and their differentiated beliefs stem from differences of opinion as in Hong and Stein (2003). The initial belief distribution Φ_0 is the prior at time 0. Depending how strong is the agents' prior, the belief distribution Φ_t is updated faster or slower (and possibly not updated for dogmatic agents). We will analyze the effects of learning.

Differentiated beliefs as differentiated "pricing measures". The second interpretation of belief is in the sense of the "pricing measure" of risk.

An agent with belief b will evaluate at time t any payoff ϕ at $t + 1$ using $\mathbb{E}[S^b \phi(Y_{t+1})] = \mathbb{E}^b[\phi(Y_{t+1})]$, where S^b is a stochastic discount factor and encodes the agents' subjective risk aversion. Using the stochastic discount factor is equivalent to discounting payoffs under the risk neutral expectation but with the modified drift b of the return (and not the real world return μ). Under the measure \mathbb{P}^b , the agent is risk averse and puts additional weight on negative outcomes. The larger the difference $\mu - b$, the larger the subjective risk aversion of the agent.

Under this interpretation, the distribution $\Phi_0 = \Phi$ may stay constant over time.

Interest rate. The lenders have as outside option a risk-free rate set to zero without loss of generality. The interest rate is constant r , and is interpreted as a spread. The firm (the firm's management) takes all credit that is provided to it and invests in the asset. The firm is assumed to act under a highly optimistic belief about the asset, so that under its belief it is optimal to invest as much funds in this strategy as possible (recall that there is no price impact in our model, just transaction costs). The equity maximization problem of the firm has been solved in the one-period case in Jarrow et al. (2016). They find that in the case of non-atomistic lenders, a sufficiently optimistic firm (e.g., a fund expecting high returns from its strategy) will place all available funds in the asset. This setting bears some resemblance to the manager in Hart and Moore (1995), whose “empire-building tendencies are sufficiently strong that it will always undertake the new investment if it can”. There, the manager's financing is constraint by the maturity structure of the debt. Here, the belief distribution will play a critical role. When sufficiently many lenders are less optimistic than the firm, the ensemble of lenders will impose a stricter constraint on leverage than the firm would set. It is this endogenous leverage imposed by lenders with heterogenous beliefs that we seek to determine.

For notational convenience, we drop the time subscript from the belief distribution Φ until we explicitly treat the learning case.

Debt capacity. Collectively the agents will choose at time t a marginal belief b_t^* and all agents with belief higher than b_t^* will invest fully and those with lower belief do not invest at all. That the agents choose collectively a strategy that is characterized by a marginal belief is quite trivial and well established in the literature. The challenge that we will face is to eliminate the multiplicity of such possible marginal beliefs, and establish b_t^* as a one-to-one map of leverage to marginal belief.

The quantity

$$\Phi(b_t^*) \tag{2}$$

is **the debt capacity at time t of one unit of the firm's asset.**

The following assumptions essentially mean that the belief distribution (across beliefs) is not heavy-tailed. It is log-convex, i.e., not more heavy than the exponential distribution. This excludes the possibility that highly optimistic lenders would concentrate a large fraction of the capital and could keep the firm liquid independently of its performance.

Assumption 2.1. *We assume that the belief distribution function $\Phi(b)$ admits a density $\phi(b)$. Moreover, we shall assume that Φ is log-convex, i.e., $b \mapsto \frac{\phi(b)}{\Phi(b)}$ is increasing on the interval $(-\infty, \Phi^{-1}(0))$. Here $\Phi^{-1}(0) \in \mathbb{R} \cup \{\infty\}$, the supremum of the support of $\phi(\cdot)$, is the maximal belief.*

Remark 2.2. *Assumption 2.1 is satisfied for a wide variety of distributions $\phi(\cdot)$, including exponential right tails, normal, or a uniform distribution. In the case with learning, this assumption must be verified at all times. We will consider the case of conjugate priors, which ensures that the updated distributions stay in the same class.*

2.2 Dynamics under given lenders' strategy

The lenders choose an **investment strategy** $\pi = (\pi_t)_{t=0, \dots, T}$ given by the functions

$$\pi_t : \mathbb{R} \times [0, \infty)^t \rightarrow [0, 1], \tag{3}$$

for $t < T$ and $\pi_T \equiv 0$, i.e., there is no investment at time T . In words, a lender with belief b lends a fraction $\pi_t(b, Y_1, \dots, Y_t)$ of her capital to the firm when the sequence of returns is (Y_1, \dots, Y_t) .

Using (1) the debt capacity at time t under investment strategy π is given by

$$D_t^\pi = \int_{\mathcal{B}} \pi_t(b) V_t^\pi \Phi(db), \quad (4)$$

with V_t^π the asset value at time t under given strategy of the lenders.

Asset value and dynamics under given strategy of the lenders. Fix an investment strategy π of the lenders. At time 0, the firm starts with the initial asset value $V_0^\pi = V_0 > 0$ and initial debt $D_{-1}^\pi = 0$. At each time $t = 0, 1, \dots, T-1$, the firm starts with the asset value V_t^π . It accepts all debt D_t^π that is offered to it, due at time $t+1$ and bearing interest rate r . The firm then pays back its maturing debt plus interest $D_{t-1}^\pi(1+r)$.

If $(1+r)D_{t-1}^\pi - D_t^\pi < 0$, then the firm has a debt inflow, which it will use to expand the asset. If $(1+r)D_{t-1}^\pi - D_t^\pi \geq 0$, then the firm has a debt outflow, and liquidates the minimum fraction of the asset in order to cover this outflow. If liquidating the entire asset cannot cover this outflow, then the firm defaults. In summary, at time t the firm needs to liquidate an amount $f_t^{-1}((1+r)D_{t-1}^\pi - D_t^\pi)$ of its asset, in order to cover a debt outflow of $(1+r)D_{t-1}^\pi - D_t^\pi$. The asset value *after the debt flow* at time t is given by

$$V_t^\pi - f_t^{-1}((1+r)D_{t-1}^\pi - D_t^\pi). \quad (5)$$

This leads to an asset value *before the debt flow* at time $t+1$ which includes the fundamental log-return $\log Y_{t+1}$

$$V_{t+1}^\pi = \left(V_t^\pi - f_t^{-1}((1+r)D_{t-1}^\pi - D_t^\pi) \right) Y_{t+1}. \quad (6)$$

We define the **default time** τ^π of the firm as the first time t when the value of its asset after the debt flow becomes negative (i.e, liquidating the entire asset cannot cover the debt outflow),

$$\begin{aligned} \tau^\pi &= \min \{t = 1, \dots, T \mid V_t^\pi - f_t^{-1}((1+r)D_{t-1}^\pi - D_t^\pi) < 0\} \\ &= \min \{t = 1, \dots, T \mid f_t(V_t^\pi) < (1+r)D_{t-1}^\pi - D_t^\pi\}. \end{aligned} \quad (7)$$

Alternatively we can write the default event in terms of the fundamental trajectory $\{\tau^\pi > t\} = \{(Y_1, \dots, Y_t) \in \Gamma_t(\pi)\}$ with

$$\Gamma_t(\pi) := \{(y_1, \dots, y_t) \mid (1+r)V_{k-1}^\pi(y_1, \dots, y_{k-1}) - V_k^\pi(y_1, \dots, y_k) \leq f_t(V_k^\pi(y_1, \dots, y_k)) \quad \forall k \leq t\}.$$

We call $\Gamma_t(\pi)$ the *survival set* of the investment strategy π .

Default relates to insolvency in a complex way⁶: At the horizon, all lenders must be paid back, so default is equivalent to insolvency. Before maturity however, the default event depends on the lenders' decisions, which in turn are based on their subjective valuation of lending to the firm (and the future default risk and recovery rate risk).

Remark 2.3. *Note that liquidation costs (different for asset purchases and asset liquidations) are not the primary source of asymmetry between inflows and outflows. Outflows will play a much more important role in the firm dynamics than inflows even in absence of any transaction costs, because default can only occur under a debt outflow scenario. We will show that asset purchase costs will not affect the long-run leverage level. This is because debt capacity is driven by default risk and in the default scenario only liquidation costs matter.*

⁶The definition of insolvency is that the value of the asset is smaller than the value of the debt.

We will refer to

$$f_t(V_t^\pi) + D_t^\pi - (1+r)D_{t-1}^\pi \quad (8)$$

as the **liquidity capacity**, and the default time can be expressed as the first time the liquidity capacity becomes 0.

On the set $\{\tau^\pi = t+1\}$, the asset is completely liquidated at time $t+1$ and the firm stops its operations. Note that $\tau^\pi = t+1$ implies $D_t^\pi > 0$ (there cannot be default in absence of debt).

Lenders' recovery rates. We assume that if the firm defaults at time $t+1$, all debt provided by *new lenders* (that is, debt provided by lenders that had not invested at time t) is immediately paid back in full by the firm, and all debt provided by *old lenders* is paid back partially as determined by the recovery rate. This assumption is not to be interpreted as seniority of the new lenders, but rather that any inflow from the new lenders can be reversed. The debt provided by old lenders, on the other hand, has been invested in the asset. So upon default, the asset position is liquidated and distributed to the old lenders. The assumption that new lenders are not diluted by using the fresh cash to increase the recovery rates of the old lenders simplifies the analysis a lot. It is also innocuous in our setting: It cannot be that the firm is in a situation of default and there is a net inflow of lenders. That would mean that there was negative performance of the asset (otherwise there would be no default) and at the same time there were more lenders than before. This is impossible under any reasonable belief updating rule.

It follows directly from (6) that on the set $\{\tau^\pi = t+1\}$, the **recovery rate** for the debt provided at time t (by the old lenders) is given by

$$f_t\left(\frac{V_{t+1}^\pi}{D_t^\pi}\right) = f\left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{D_t^\pi} - 1\right)\right)Y_{t+1}. \quad (9)$$

Finally we assume that at time T , all debt must be payed back, that is, $\pi_T(\cdot, \cdot) \equiv 0$ and $D_T^\pi = 0$.

Remark 2.4. It follows recursively from $V_0^\pi = V_0$ and plugging (4) into (6) that the dependence of V_t^π on π is only through π_0, \dots, π_{t-1} .

A key quantity in the subsequent analysis will be played by the firm's **leverage (or debt-to-asset) ratio** before the debt flow

$$X_t^\pi = \frac{D_{t-1}^\pi}{V_t^\pi}, \quad t = 1, \dots, T \wedge \tau^\pi,$$

with the convention $X_t^\pi = \infty$ for $t > \tau^\pi$.

Below, we introduce rigorously the lender game and the payoffs. It is worth summarizing at this point the sequence of actions and point out the inter-temporal entanglement of the lenders' decisions that we will study in the sequel, see Figure 2. In equilibrium, the strategy will have a marginal lender b_t^* , to be determined as a function of debt-to-asset.

2.3 Lenders' game

The lenders choose the strategy $(\pi_t)_{t=0, \dots, T}$ in (3). Note in particular that all lenders with the same belief play the same strategy, so we classify lenders according to their beliefs and we refer to lender b to any lender with belief b .

Payoff functions. Let π be a strategy and $t \leq \tau^\pi$. We denote by $G_t^{\pi, b}$ the payoff of lender b under strategy π . The payoff has two branches.

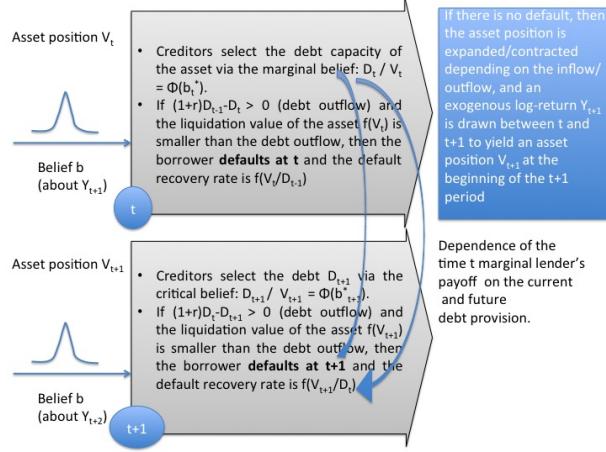


Figure 2: Sequence of actions and inter-temporal entanglement (arrows point to the dependence of current payoffs on current and future debt provision).

- On $\{\tau^\pi > t\}$ (survival), lender b 's expected return per dollar from lending at time t is⁷

$$\begin{aligned} R_t^{\pi,b} &:= \mathbb{E}^b \left[\mathbf{1}_{\{\tau^\pi > t+1\}} (1+r) + \mathbf{1}_{\{\tau^\pi = t+1\}} f_t \left(\frac{V_{t+1}^\pi}{D_t^\pi} \right) \mid (Y_1, \dots, Y_t) \right] - 1 \\ &= r - \mathbb{E}^b \left[\mathbf{1}_{\{\tau^\pi = t+1\}} (1+r - f_t \left(\frac{V_{t+1}^\pi}{D_t^\pi} \right)) \mid (Y_1, \dots, Y_t) \right], \end{aligned} \quad (10)$$

and the payoff at time t is $\pi_t(b, Y_1, \dots, Y_t) R_t^{\pi,b}$.

- On the set $\{\tau^\pi = t\}$ (default), the payoff (per dollar invested) of any old lender at time t is given by $f_t \left(\frac{V_t^\pi}{D_{t-1}^\pi} \right) - 1$.

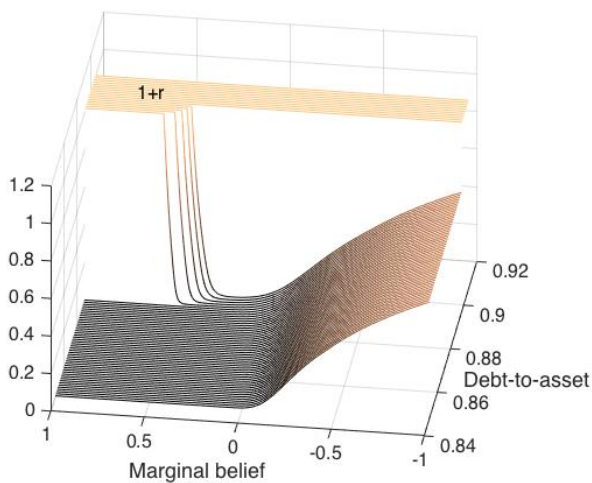
Tradeoff and the lenders' short put option. The expected return per dollar invested highlights the tradeoff of a lender with belief b . On one side, she earns the interest r . On the other side, she bears the expected default cost. Her default cost depends on the other lenders' decisions both in the current period (the asset would be divided by the total amount of debt in case of default), and in the next period (because they determine the default time). The dependence on the belief is highly non-linear, and any lender sees a strictly positive expected default cost. In fact, lenders' investment is akin them being short a put option on the (risky) recovery rate $f_t \left(\frac{V_{t+1}^\pi}{D_t^\pi} \right)$. The higher their belief, the lower their expected default cost.

At the core of our paper is the valuation of the put option at a marginal belief. The higher the valuation, the worse the payoff because lenders are short this option. Figure 3 plots the put option surface as a function of the marginal belief and of the firms' leverage. Here we are interested in a preview of the monotonicity properties, the specifics of the option valuation are detailed below in Proposition 3.3 and the analysis is in Appendix A.2.

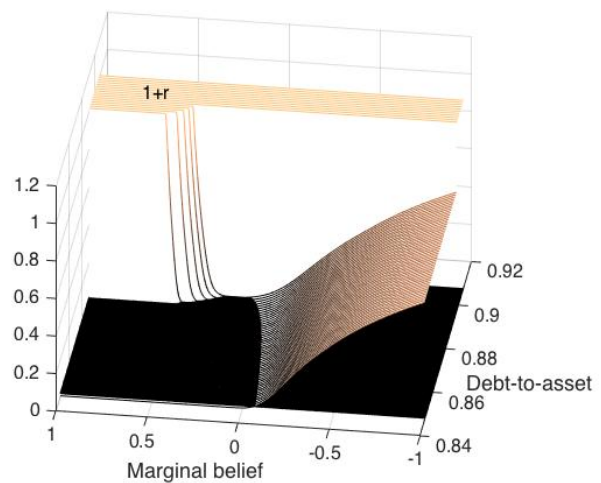
The parameters we used for the plots are typical to those we use in the numerical results section.⁸ Note that when the marginal belief is too high and the firm too leveraged, then the firm cannot survive (because the capital above that marginal belief is insufficient). In this case the put option is valued at the maximum $1+r$ and the lender's payoff is $r - (1+r) = -1$, i.e., she expects full loss of her capital.

⁷Note that $R_t^{\pi,b}$ is a deterministic function of (Y_1, \dots, Y_t) .

⁸The valuation is reminiscent of Merton's model: the debt-holders are short a put option on the firms' assets. Here the lenders have differentiated values of this put option.



(a) Value of the put option



(b) The marginal lender is such that her value of the put option is equal to r (the intersecting plane) ($r = 1\%$ in this example).

Figure 3: (Preview) Value of the put option by a marginal lender as a function of the marginal belief and the firm's current leverage. For high leverage, payoffs are non-monotonous as a function of the marginal belief. This monotonicity are shown rigorously using option valuation techniques, see Proposition 3.3, the analysis in the Appendix A.2.

Figure 3b shows how the marginal lender is determined in equilibrium by equating the put option value with the outside option r . The curve where the option surface intersects the plane gives the marginal lender's belief as a function of the firm's leverage. This curve is further detailed in Figure 4.

Figure 4 illustrates that for high enough values of the debt-to-asset there are two possible marginal lenders (and correspondingly two possible debt capacity values). This is a result of the non-monotonous relation between belief and the value of the put option as shown in Figure 3a. The detailed analysis in Section 3 will demonstrate that only the one with the highest debt is a viable equilibrium.

Pessimistic lenders transform a sequence of mild negative returns into large liquidity shocks. Consider a sequence of mild negative returns. Because of the convexity illustrated in Figure 4, the marginal lender will increase fast in response to the negative returns (which increase the leverage). The convexity effect is more pronounced when the leverage is high. High leverage is attained when pessimistic lenders are drawn in, for example after when asset returns are high and the firm builds equity. However, pessimists are the fastest to withdraw because the value of their put option (and expected default costs) increase most under negative returns. This leads to spiraling effects and a predictable collapse in debt capacity. The spiraling effects also depend on our belief distribution assumption: after a sequence of withdrawals only the highly optimistic lenders stay, and they do not concentrate a large fraction of the capital mass.

Spiraling effects happen when the dependence of the marginal belief on leverage is stronger (higher convexity). When this dependence is weaker (lower convexity), then the firm will be able to recover from a sequence of negative returns. Deleveraging is not so pronounced in this case: the marginal belief does not change too much, and the outflows are sufficiently small. At the same time, lower convexity comes along with lower leverage and a higher equity ratio. Combined, these

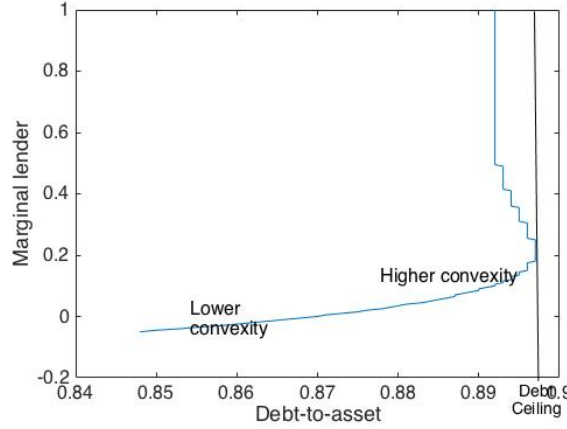


Figure 4: (Preview) Firm's leverage and marginal lender's belief. For high leverage, there are two marginal lenders corresponding to two equilibria. The rigorous analysis of the monotonicity under different regimes of leverage is given in Appendix A.2, see Corollary A.2.

two effects allow the firm to absorb the deleveraging. These weaker and stronger convexity effects concur to the existence of a mean-reverting and an explosive leverage, that we will demonstrate in Section 4.

In contrast to debt financing, with equity financing only optimists invest. Moreover, since there is no put option, a model with equity financing would behave differently: no spiraling effects would happen unless we introduce a shift in the belief distribution, or asymmetric information that is revealed through returns.

Equilibrium definition. In a Nash equilibrium, no lender can increase her payoff by changing her strategies, if the other lenders keep theirs.

Definition 2.5 (Nash equilibrium). *By convention, any strategy is a Nash equilibrium at time T . A strategy π^* is called a Nash equilibrium at time $t < T$ if for each belief $b \in \mathcal{B}$ and each strategy π which satisfies $\pi_s = \pi_s^*$ for all $s < t$ and $\pi_t(b') = \pi_t^*(b')$ for $b' \neq b$ we have $G_t^{\pi^*,b} \geq G_t^{\pi,b}$. A strategy is said to be Nash equilibrium (equilibrium for short) if it is a Nash equilibrium at all times.*

We now consider the two branches (survival and default) of the payoff G_t . Because the lenders are infinitesimally small, a unilateral deviation of lender b at time t cannot change if the firm defaults or not at that time and nor can it change b 's payoff in the case of default (determined by the recovery rate). Therefore, any strategy which leads to default is automatically a Nash equilibrium. For strategies that do not lead to default, the Nash equilibrium condition translates into a cutoff property for the expected return R_t : lenders invest if and only if their expected return is positive. We therefore have the following equivalent definition of Nash equilibria.

Definition 2.6 (Nash equilibrium characterization). *A strategy π^* is called a Nash equilibrium at time $t < T$ if for every fundamental trajectory (Y_1, \dots, Y_t) we have that either:*

- $(Y_1, \dots, Y_t) \notin \Gamma_t(\pi^*)$ (the firm defaults at time t) , or
- $(Y_1, \dots, Y_t) \in \Gamma_t(\pi^*)$ (the firm survives at time t) and for each belief $b \in \mathcal{B}$

$$\pi_t^*(b, Y_1, \dots, Y_t) = \begin{cases} 1 & \text{if } R_t^{\pi^*,b} > 0, \\ 0 & \text{if } R_t^{\pi^*,b} < 0. \end{cases} \quad (11)$$

Remark 2.7. If π^* is a Nash equilibrium and $(Y_1, \dots, Y_t) \in \Gamma_t(\pi^*)$, then $D_t^{\pi^*} > 0$. Suppose $D_t^{\pi^*} = 0$. Then there cannot be default at time $t + 1$, so $R_t^{\pi^*, b} = r > 0$, for all b . Therefore, all lenders would invest, in contradiction to $D_t^{\pi^*} = 0$.

As soon as the firm attains high enough leverage, there are infinitely many Nash equilibria in which the firm defaults. For example $\pi_t \equiv 0$ pushes the firm to default at time t as soon as $D_{t-1}(1+r) - f_t(V_t) > 0$.⁹ In this case, there is an infinity of strategies π_t with $\int_{\mathcal{B}} \pi_t(b) V_t^\pi \Phi(db) < D_{t-1}(1+r) - f_t(V_t)$, and all these are Nash equilibria which lead to default.¹⁰

If the Nash equilibria in which the firm defaults coexist with Nash equilibria in which the firm survives, then these equilibria with default can be eliminated by removing weakly dominated strategies as in Krishenik et al. (2015), see Appendix A.5 for the definition and intuition behind weakly dominated strategies. We assume that lenders will not play weakly dominated strategies at time t . In contrast to Krishenik et al. (2015), elimination of dominated strategies is not sufficient here to prove uniqueness because there may be two Nash equilibria in which the firm survives. This is due to the presence of recovery rates, which induce the non-monotonicity in the expected return as a function of the belief of the marginal lender, see Figures 3b and 4.

We extend the solution concept “Strongly coalition proof” introduced by Milgrom and Roberts (1996). Such an equilibrium is proof to any deviations by coalitions which are stable in the sense that they are Nash equilibria themselves all else fixed outside the coalition. We will show in Theorem 3.6 that the first Nash equilibrium (with higher debt) is “Strongly coalition proof”. The “Strongly coalition proof” is a stronger requirement than “Coalition proof” in the sense of Bernheim et al. (1987), see the discussion in Milgrom and Roberts (1996). Therefore the first Nash equilibrium is automatically “Coalition proof”.

We introduce the weaker condition “Strongly ϵ -coalition proofness”, in which we require stability with respect to deviations of coalitions that are Nash equilibria and in addition are arbitrarily small. If an equilibrium is not “Strongly ϵ -coalition proof”, it automatically implies that is not “Strongly coalition proof”. We will show in Theorem 3.7 that the Nash equilibrium with lower debt is not “Strongly ϵ -coalition proof” for any $\epsilon > 0$ (and hence it is not “Strongly coalition proof”). Note the importance of “for all $\epsilon > 0$ ”: it would be easy to show that a large group of lenders can be better off by jointly deviating from the second equilibrium. But our results are much stronger: *no matter how small (but positive) the size*, we can always find a stable lender coalition of that small size that can be better off by jointly deviating from the second equilibrium.¹¹ Therefore it is reasonable to exclude this second equilibrium.

Definition 2.8 (Strongly ϵ -coalition proofness). Let $\epsilon > 0$. A Nash equilibrium π^* is said to be **strongly ϵ -coalition proof** at time T by convention, and at time $t < T$ if:

For each interval of lenders $[b_0, b_1] \subseteq \mathcal{B}$ with $b_1 - b_0 < \epsilon$ and each strategy π which satisfies

1. $\pi_s(\cdot, \cdot) = \pi_s^*(\cdot, \cdot)$ for all $s \neq t$
2. $\pi_t(b', \cdot) = \pi_t^*(b', \cdot)$ for all $b' \in \mathcal{B} \setminus [b_0, b_1]$

⁹This condition means that the early liquidation of the entire asset cannot cover existing debt plus interest, which is the typical case of a leveraged firm.

¹⁰Note that there may be infinitely many strategies π with survival, i.e., $\int_{\mathcal{B}} \pi_t(b) V_t^\pi \Phi(db) \geq D_{t-1}(1+r) - f_t(V_t)$, but these are not Nash equilibria in general. Only if one finds a marginal lender whose value of the put option is equal to r (see Figure 3b), then we have a Nash equilibrium.

¹¹For a game with a continuum of players it is very natural to consider coalitions of an arbitrarily small but positive fraction $\epsilon > 0$ of all lenders: a player has zero mass in the continuum limit, whereas in any approximating game with finite but large number of players she would have a small but positive mass. Convergence results for finite games in the spirit of Carmona et al. (2017) are left for future research.

3. (Stability of the coalition: all else fixed, the coalition plays a Nash equilibrium)

$$\pi_t(b, Y_1, \dots, Y_t) = \begin{cases} 1 & \text{if } R_t^{\pi, b} > 0, \\ 0 & \text{if } R_t^{\pi, b} < 0, \end{cases} \text{ for each } b \in [b_0, b_1] \text{ and } (Y_1, \dots, Y_t) \in \Gamma_t(\pi), \quad (12)$$

we have that the coalition is “worse off”:

a) $\Gamma_t(\pi) \subseteq \Gamma_t(\pi^*)$ (the survival set decreases), and

b) $R_t^{\pi^*, b} \geq R_t^{\pi, b}$ for all $(Y_1, \dots, Y_t) \in \Gamma_t(\pi)$ and $b \in [b_0, b_1]$ (return decrease on trajectories with survival).

A strategy is called Strongly ϵ -coalition proof if it is Strongly ϵ -coalition proof at all times t .

The solution concept strongly coalition proof is the same as strongly ∞ -coalition proof.

3 Nash equilibria for the lenders' game

The goal of this section is to determine the set of equilibria of the lenders' game. That Nash equilibria in which the firm survives are of cut-off type is straightforward (see Lemma A.3).

The subtlety is to show that the marginal belief is uniquely determined as a function of the firm's debt-to-asset ratio X_t^π . We will also need to show that the survival set $\Gamma_t(\pi)$ can be also expressed in terms of the debt-to-asset. As already previewed in Figure 4, there are two possible marginal belief functions. This will be proved rigorously in Section 3.1 below. Note that in this case the number of corresponding equilibria grows exponentially with the number of periods: one marginal belief function tomorrow yields two possible marginal belief functions today, and so on. We will prove uniqueness of an ϵ -coalition proof equilibrium and only then the leverage X_t^π will be a proper state variable.

Definition 3.1 (Strategy with marginal belief function). A strategy π is said to have the marginal belief function $\beta_t(\cdot) : [0, \infty] \rightarrow \mathcal{B}$ at time t if

$$\pi_t(b) = \mathbf{1}_{\{b \geq \beta_t(X_t^\pi)\}} \text{ on } \Gamma_t(\pi). \quad (13)$$

If a Nash equilibrium has a marginal belief function $\beta_t(\cdot)$ at time t , then its corresponding marginal belief function $\beta_t(\cdot)$ satisfies

$$R_t^{\pi, \beta_t(X_t^\pi)} = 0. \quad (14)$$

Let now π be a strategy with marginal belief function $\beta_t(\cdot)$ at time t as in (13). By (4), the debt of the firm at time t on the set $\{\tau^\pi > t\}$ is given by

$$D_t^\pi = V_t^\pi \Phi(\beta_t(X_t^\pi)), \quad (15)$$

which is strictly positive. Recall that the liquidity capacity at time t of the firm is given by (8). On the set $\{\tau^\pi > t\}$, the ratio of the liquidity capacity of the firm and the asset can be written as $\lambda_t(X_t^\pi)$ as defined below.

Definition 3.2 (Liquidity capacity function). Let $\lambda_t : [0, \infty] \rightarrow [0, \infty]$ defined as

$$\lambda_t(x) := 1 - \alpha \mathbf{1}_{\{t < T\}} + \Phi(\beta_t(x)) - (1 + r)x. \quad (16)$$

It now follows from (7) that

$$\tau^\pi > t \Rightarrow \lambda_t(X_t^\pi) \geq 0. \quad (17)$$

The converse is not *a priori* true since on the default set $\{\tau^\pi = t\}$, the new debt is not given by (15)¹². We will show below that we do have equivalence and the function λ_t is uniquely determined if π is a ϵ -coalition proof Nash equilibrium in which agents do not use weakly dominated strategies. The proof will be by backward induction and will use the following results as a building block.

3.1 One-period building block for the equilibrium

In this section we give explicit formulas for the lenders' expected return R_t^π , akin those of option prices in the Black and Scholes model. These formulas allow us to study analytically the monotonicity (at time t) of the expected return of the marginal lender as a function of her belief, that we illustrated in Figure 4. This analysis will be part of the induction step in our proof of the uniqueness theorem 3.7.

Assume¹³ that we have equivalence in (17) for time $t + 1$, i.e.,

$$\tau^\pi = t + 1 \Leftrightarrow \lambda_{t+1}(X_{t+1}^\pi) < 0 \quad (18)$$

and, moreover, that the marginal belief function $\beta_t(\cdot)$ is increasing and consequently $\lambda_t(\cdot)$ is decreasing. Then, on the set $\{\tau^\pi > t\}$, by (6) we have that the default event can be expressed as the discounted asset return being below a *discounted strike* k_t , where the discount rate is equal to the belief variable b

$$\begin{aligned} \tau^\pi = t + 1 &\Leftrightarrow \lambda_{t+1}\left(\frac{D_t^\pi}{V_{t+1}^\pi}\right) < 0, \\ &\Leftrightarrow \frac{D_t^\pi}{V_{t+1}^\pi} > \lambda_{t+1}^{-1}(0), \\ &\Leftrightarrow \frac{D_t^\pi}{V_t^\pi \left(1 - f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{V_t^\pi} - \frac{D_t^\pi}{V_t^\pi}\right)\right) Y_{t+1}} > \lambda_{t+1}^{-1}(0), \\ &\Leftrightarrow Y_{t+1} e^{-b} < \frac{e^{-b} \frac{D_t^\pi}{V_t^\pi}}{\lambda_{t+1}^{-1}(0) \left(1 - f_t^{-1}\left((1+r)\frac{D_{t-1}^\pi}{V_t^\pi} - \frac{D_t^\pi}{V_t^\pi}\right)\right)} =: k_t\left(b, \frac{D_t^\pi}{V_t^\pi}, \frac{D_{t-1}^\pi}{V_t^\pi}\right). \end{aligned} \quad (19)$$

Since we have log-normal returns, we can use the Black and Scholes machinery to express the expected return (similar to a short put position) as a function of X_t^π .

Proposition 3.3. *Let π a strategy with a marginal belief function $\beta_{t+1}(\cdot)$ at time $t + 1$, for some $t < T$ which satisfies (18). Then on the set $\{\tau^\pi > t\}$ the expected return under belief b satisfies*

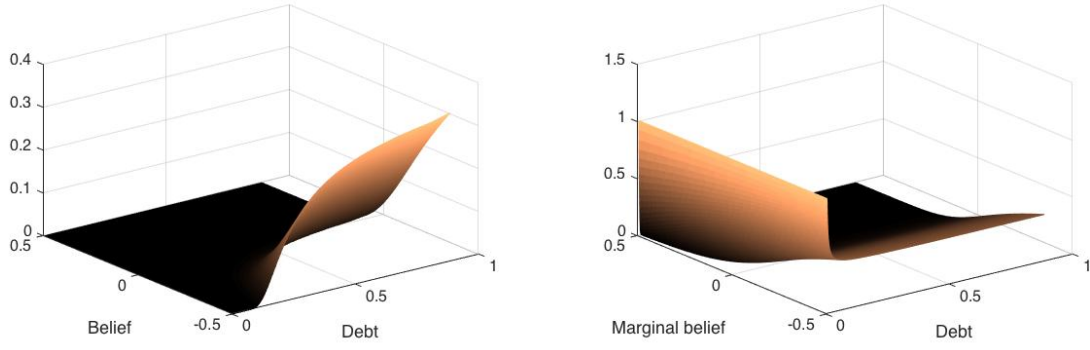
$$R_t^{\pi, b} = h_t\left(k_t\left(b, \frac{D_t^\pi}{V_t^\pi}, X_t^\pi\right)\right) \quad (20)$$

with

$$\begin{aligned} h_t(K) &= r - (1+r)\mathcal{N}(-d_2(K)) + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)K}\mathcal{N}(-d_1(K)), \\ d_{1,2}(K) &= \frac{-\log K \pm \frac{1}{2}\sigma^2}{\sigma}. \end{aligned}$$

¹²The firm's equity typically cannot cover the liquidation costs for its entire asset. In such case the strategy in which no one lends $\pi \equiv 0$ is a trivial Nash equilibrium. However, if the debt-to-asset is not too large, such equilibrium will not survive the elimination of weakly dominated strategies.

¹³These assumptions on the monotonicity properties at time $t + 1$ will be part of the induction hypothesis in the proof of the uniqueness theorem.



(a) Put option surface (debt-to-asset = 0.8) shows strategic substitutability. (b) Put option surface (debt-to-asset = 0.9) shows strategic complementarity

Figure 5: Lenders' put option value as a function of belief and concurrent debt (expressed as percentage of lenders' capital). Payoffs are given by r minus the value of the put option. When current leverage is lower (left), pessimists's put option value increases (and their payoff decreases) with the amount of concurrent debt. When current leverage is higher (right), pessimists's put option value decreases (and their payoff increases) with the amount of concurrent debt up to a certain point.

Suppose moreover that $\beta_{t+1}(\cdot)$ is strictly increasing. Then the function $h_t(K)$ is strictly decreasing in $K > 0$ and satisfies $\lim_{K \rightarrow 0} h_t(K) = r$ and $\lim_{K \rightarrow \infty} h_t(K) = -1$.

By (20) and (15), the expected return at time t on the set $\{\tau^\pi > t\}$ under belief b for a strategy π as in Proposition 3.3 is given by

$$R_t^{\pi, b} = h_t\left(k_t(b, \Phi(\beta_t(X_t^\pi)), X_t^\pi)\right),$$

which is clearly increasing¹⁴ with b .

Strategic substitutability and complementarity. Having established the expected return, we are ready to plot the payoffs that arise in our leverage/deleveraging game, namely we plot in Figure 5 the surface $(b, \Phi) \rightarrow h_t(k_t(b, \Phi, X_t))$ for two different values of the current leverage X_t . These plots illustrate the complex dependencies in our payoff structure. There is a change from global strategic substitutability to one-sided strategic complementarity as leverage increases. The sources of these properties are novel in the literature, as discussed in the introduction.

3.2 Existence and uniqueness results

In this section we show that there exists a strongly coalition proof equilibrium, namely the strategy $\hat{\pi}$ given in Definition 3.4. We also show that any other Nash equilibrium in which the firm would survive with lower debt is not proof to arbitrarily small deviations, in the sense that *no matter how small (but positive) the size*, we can always find a stable lender coalition of that size that can be better off by jointly deviating from that equilibrium.

¹⁴This is not to be confused with $b \rightarrow h_t(k_t(b, \Phi(b), X_t^\pi))$ which is the return under a *marginal belief* and is non-monotonous, as in Figure 3.

From Proposition 3.3, the equation (14) takes the form

$$h_t(k_t(\beta_t(X_t^\pi), \Phi(\beta_t(X_t^\pi)), X_t^\pi)) = 0. \quad (21)$$

In Appendix A.2 we show that this equation will have two solutions in the marginal belief function β_t . In the proof of Theorem 3.7, we will show that the larger solution cannot correspond to the marginal belief function in an ϵ -coalition proof equilibrium. Indeed, at the larger solution, the marginal lender is optimistic about the firm's asset and higher debt will increase her expected payoff, since the firm will expand the asset. Therefore, slightly less optimistic potential lenders than the marginal one will join a coalition and invest, leading to positive returns for themselves and the former marginal one.

On the other hand, the equilibrium corresponding to the smaller solution of the equation, denoted $\hat{\beta}_t$, is not only ϵ -coalition proof, but coalition proof for any coalition size. This equilibrium is defined by the marginal belief functions $\hat{\beta}_t$, which satisfy the assumptions of Proposition 3.3 and (21).

Definition 3.4. *a) Let $\hat{\pi}$ the equilibrium with marginal belief functions $\hat{\beta}_t(\cdot)$, defined backward recursively for $t = T, \dots, 0$ as follows. Set $\hat{\beta}_T(\cdot) \equiv \infty$. Given the function $\hat{\beta}_{t+1}(\cdot)$ for $t < T$, let*

$$\begin{aligned} \hat{\lambda}_{t+1}(x) &= 1 - \alpha \mathbf{1}_{\{t+1 < T\}} + \Phi(\hat{\beta}_{t+1}(x)) - (1+r)x, \\ \hat{h}_t(K) &= r - (1+r)\mathcal{N}(-d_2(K)) + \frac{1-\alpha}{\hat{\lambda}_{t+1}^{-1}(0)K} \mathcal{N}(-d_1(K)), \\ \hat{k}_t(b, q, x) &= \frac{e^{-bq}}{\hat{\lambda}_{t+1}^{-1}(0) \left(1 - f_t^{-1}((1+r)x - q)\right)}, \end{aligned}$$

and then let $\hat{\beta}_t(x)$ denote the **smallest** solution of the equation

$$\hat{h}_t(\hat{k}_t(\hat{\beta}_t(x), \Phi(\hat{\beta}_t(x)), x)) = 0 \quad (22)$$

for all $x \geq 0$ for which there exists a solution, and $\hat{\beta}_t(x) = \infty$ otherwise.

The definition of the previous functions uses tacitly that $\hat{\lambda}_{t+1}^{-1}(0)$ is well defined. This is indeed the case, by virtue of the following result.

Proposition 3.5. *The function $\hat{\beta}_t(\cdot)$ is increasing. Moreover, under the condition*

$$\phi'(\cdot) \leq \Phi(\cdot) + 2\phi(\cdot) \quad (23)$$

the function $\hat{\beta}_t(\cdot)$ is strictly convex.

Together with the log-concavity assumption, we have the condition $-\frac{\phi(\cdot)^2}{\Phi(\cdot)} \leq \phi'(\cdot) \leq \Phi(\cdot) + 2\phi(\cdot)$. Condition (23) is technical and allows us to control the left tail of the belief distribution. It is for example satisfied for the normal left tails, and also for any distribution in which $\phi' \leq 0$, such as the case when the density increases with the pessimism level. While the log-concavity controls the right tails (the capital of the optimists should decrease fast enough), the control of the left tails ensures on the contrary that the pessimists hold most of the capital.

Notation. In the sequel, we denote by $\pi^{(s)}$ any strategy which will have the marginal belief functions $\hat{\beta}_t(\cdot)$ from time s on, $t \geq s$ and its survival sets are given by

$$\Gamma_t(\pi^{(s)}) = \{y_t \mid \hat{\lambda}_t(X_t^{\pi^{(s)}}(y_t)) \geq 0\}. \quad (24)$$

In particular, we have $\pi^{(0)} = \hat{\pi}$. This strategy is uniquely determined from time s on, while it is a generic strategy before time s . Any strategy satisfying these conditions is said to be *of the form* $\pi^{(s)}$. It is immediate to see that this strategy is a Nash equilibrium, so the proof is mainly focused on strongly coalition proofness.

Theorem 3.6 (Existence). *Any strategy of form $\pi^{(t)}$ is a strongly coalition proof Nash equilibrium at time t . In particular $\hat{\pi}$ is a strongly coalition proof Nash equilibrium.*

The following uniqueness result states that any other Nash equilibrium than $\hat{\pi}$ can be blocked by deviations of arbitrarily small groups of lenders, i.e., it is not strongly ϵ -coalition proof for any $\epsilon > 0$ (and in particular this makes $\hat{\pi}$ the unique strongly coalition proof equilibrium).

Theorem 3.7 (Uniqueness). *Let π be a Strongly ϵ -coalition proof Nash equilibrium for an $\epsilon > 0$ and assume that no agent uses weakly dominated strategies. Then we have that $\pi = \hat{\pi}$.*

Our existence and uniqueness results have critical implications for leverage stability. Indeed, because the second Nash equilibrium (with lower debt) can be blocked by any small stable coalition it can be reasonably excluded, and lenders will select the equilibrium that gives the firm the higher leverage. The convexity effects of the marginal belief as function of leverage are properties of the equilibrium $\hat{\pi}$ and underlie the existence of explosive regimes of leverage that we discuss in Section 4.

The proof of the uniqueness result is by induction: we show successively that $\pi = \pi^{(t+1)}$ leads to $\pi = \pi^{(t)}$, for all $t = T - 1, \dots, 0$ and thus we uniquely identify π as the equilibrium with marginal belief $\hat{\beta}$ at all times, namely we have that $\pi = \hat{\pi}$.

We rely on the one period building block in the previous section. At step $s < T$ we use the induction hypothesis that $\pi = \pi^{(t+1)}$. The one period building block gives analytical formulas for the expected return as a function of the belief of the marginal lender, conditional on survival at time t . Using the elimination of weakly dominated strategies, we can establish if the fundamental trajectory y_t leads to default or not: the firm survives if and only if there exists a marginal lender with zero expected return. Technically, this means that there is a solution to equation (22).

However, conditional on survival there may be two marginal lenders with zero expected return, and they correspond to the solutions of the equation (22). Note the necessity of using backward induction: assuming that we have removed uncertainty about the future strategies, we can use equation (22) to define a Nash equilibrium in the current period. Using the ϵ -coalition proof refinement, we eliminate the largest solution to the equation in the current period, and complete the induction step since the smallest equation gives the belief function $\hat{\beta}_t$ which defines the strategy $\pi^{(t)}$.

The proof relies on many results that fit together in a highly complex structure. As a final detail of the complex structure, we note that the assumptions of Proposition 3.3 in the building block hold due to the induction hypothesis. When $\pi = \pi^{t+1}$, by virtue of Proposition 3.5, we have that $\hat{\beta}_{t+1}(\cdot)$ is increasing.

3.3 Ceiling

By eliminating the multiplicity of equilibria, we eliminate the uncertainty about the value of the ceiling, since each equilibrium has its own ceiling associated to it, see Figure 6. With multiple equilibria, default cannot be determined by the value of the debt-to-asset.

This unique equilibrium is characterized by the highest value for the ceiling. Removing uncertainty about the ceiling means that default becomes measurable with respect to the observation

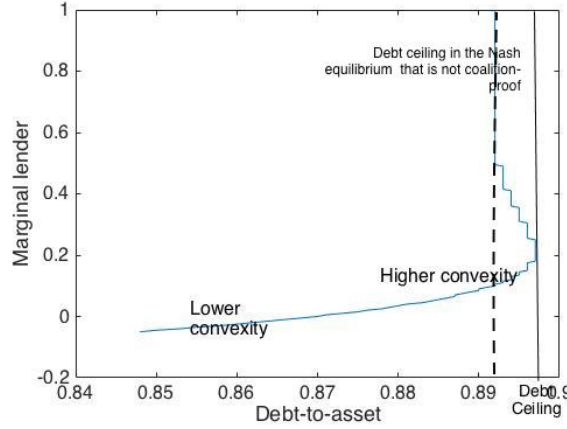


Figure 6: The debt ceiling is associated to the equilibrium $\hat{\pi}$, with belief function given in Definition 3.4 as the smallest solution to (22). A second equilibrium has belief function given by the largest solution to (22). When the debt-to-asset is at the lower ceiling, there is uncertainty about which equilibrium will be selected and about the firm's survival (it would survive under equilibrium one, but not under equilibrium two).

of the debt-to-asset. Indeed, from (24) it now follows that the company defaults as soon as the debt-to-asset ratio $X_t = \frac{D_t - 1}{V_t}$ exceeds the value $\hat{\lambda}_t^{-1}(0)$, which henceforth we shall refer to as the firm's *ceiling*. The next result shows that the ceiling can be written explicitly in terms of a suitable deterministic function $m_t(\cdot)$ given in equation (38) in the appendix.

Proposition 3.8. *The ceiling for the debt-to-asset ratio is given by*

$$\hat{\lambda}_t^{-1}(0) = m_t^{-1}\left(\frac{1}{\hat{h}_t^{-1}(0)}\right). \quad (25)$$

The debt ceiling gives the default condition. In the next section, we determine the regimes of leverage. We also give a measure of stability when the debt is in the mean reverting regime.

4 Debt dynamics and debt stability

For the remainder of the paper, we assume that lenders select the unique strongly coalition proof equilibrium $\hat{\pi}$ at all times $t = 0, 1, \dots, T - 1$ and we drop the superscript $\hat{\pi}$ from our notation of the processes D , V and X . We also drop the hat from the functions $\hat{\lambda}_t(\cdot)$, $\hat{h}_t(\cdot)$ and $\hat{k}_t(\cdot)$ in Definition 3.4. The debt-to-asset is a state variable, whose dynamics and stability can be characterized.

We consider that the debt-to-asset is below the debt ceiling that we established in the previous period, i.e., $X_t \leq \lambda_t^{-1}(0)$, and we characterize its dynamics. At the debt ceiling, the firm defaults. We can write the debt-to-asset ratio in the next period as a function of the debt-to-asset ratio in

the current period, starting from (6):

$$\begin{aligned}
X_{t+1} &= \frac{D_t}{V_t \left(1 - f_t^{-1} \left((1+r) \frac{D_{t-1}}{V_t} - \frac{D_t}{V_t} \right) \right) Y_{t+1}} \\
&= \frac{\lambda_{t+1}^{-1}(0) k_t \left(\beta_t(X_t), \frac{D_t}{V_t}, \frac{D_{t-1}}{V_t} \right) e^{\beta_t(X_t)}}{Y_{t+1}} \\
&= \lambda_{t+1}^{-1}(0) h_t^{-1}(0) e^{\beta_t(X_t)} \frac{1}{Y_{t+1}}, \tag{26}
\end{aligned}$$

where in the second line we used the definition of k_t in Definition 3.4 and (15), and in the last line we used (43).

Our notion of stability is that there exists a level x_t strictly below the debt ceiling $\lambda_t^{-1}(0)$, such that the process X_t is mean-reverting to the level x_t under *the real-world measure* \mathbb{P} as long as the process stays below an instability level \bar{x}_t , with $x_t < \bar{x}_t \leq \lambda_t^{-1}(0)$.

Stability measure. Of course, since our model is in discrete time, the process X_t can jump from the long-run level over the instability level in one period. The probability of such jump is a necessary addendum to characterization of the stability of the process in terms of the various regimes. We define the stability measure (for debt with a mean-reverting regime) as a (conditional) probability to stay below the instability level \bar{x}_t in one period if one forces leverage to start the long-run level x_t :

$$1 - \mathbb{P}[X_{t+1} > \bar{x}_t \mid X_t = x_t]$$

We define this measure using a conditional probability because we think of the assessment of the entire mean reverting regime. If the stability measure is too low, then even if the process is in the mean-reverting regime, it can very easily "escape" in the explosive regime. Clearly, the stability measure increases the difference between the instability level and the long run level. The existence of the mean-reverting regime and the stability measure together determine the sustainability of debt.

Definition 4.1. Let $p \in [0, 1]$. We say that *debt-to-asset has a mean-reverting regime* $[0, \bar{x}_t)$ (with stability measure $1 - p$) if for each t there exist $0 < x_t < \bar{x}_t \leq \lambda_t^{-1}(0)$ such that

$$\begin{aligned}
\mathbb{E}[X_{t+1} \mid (Y_1, \dots, Y_t)] &> X_t, \forall X_t \in (0, x_t) \cup (\bar{x}_t, \lambda_t^{-1}(0)) \\
\mathbb{E}[X_{t+1} \mid (Y_1, \dots, Y_t)] &< X_t, \forall X_t \in (x_t, \bar{x}_t).
\end{aligned}$$

and $\mathbb{P}[X_{t+1} > \bar{x}_t \mid X_t = x_t] < p$. We call x_t *the long-run level*, \bar{x}_t *the instability level* and $(x_t, \lambda_t^{-1}(0))$ *the explosive regime*. If *debt-to-asset does not have a mean-reverting regime*, then it is said to be *explosive*.

In the mean-reverting regime (if any) X_t reverts to the long-run level x_t as long as it stays below the instability level \bar{x}_t . Clearly, the definition uses the expected return of the asset under the real-world ("oracle") measure. We now let the real-world expected return $e^\mu = \mathbb{E}[Y_t]$. Using the dynamics of X_t above, we can compute the *drift function* of the process X_t as

$$a_t(X_t) := \mathbb{E}[X_{t+1} \mid (Y_1, \dots, Y_t)] = \lambda_{t+1}^{-1}(0) h_t^{-1}(0) e^{\beta_t(X_t)} e^{-\mu + \sigma^2}. \tag{27}$$

Remark 4.2 (Drift invariance under the belief of the marginal lender). *Similarly to computing the drift of the process X_t under the real world measure, we can compute the drift under the belief*

of the marginal lender. At any time before default, the marginal lender's expectation of the future debt-to-asset is constant and given by

$$\mathbb{E}^{\beta_t(X_t)}[X_{t+1} | (Y_1, \dots, Y_t)] = \lambda_{t+1}^{-1}(0)h_t^{-1}(0)e^{\sigma^2}.$$

The continuum of lenders adjust debt in each period such that the expectation of the future leverage from the perspective of the marginal lender is invariant. In turn, a marginal lender exists (and there is no default) as long as this adjustment of debt is possible.

This also explains why the instability level \bar{x}_t can be different from the debt ceiling $\lambda_t^{-1}(0)$. Indeed, when debt-to-asset reaches the instability level an "oracle" can detect that the process is explosive and expected to reach the debt ceiling. But the default does not happen yet, because a marginal lender can be found and under her belief the debt-to-asset is not explosive yet. As we will later show, when agents learn, then the instability level and the debt ceiling coincide: $\bar{x}_t = \lambda_t^{-1}(0)$. In this case, the expectation of any lender becomes close to the "oracle"'s expectation. Therefore, all lenders can detect when the debt-to-asset becomes explosive and default happens at this point.

We now assume that the belief distribution function satisfies (23). Then by Proposition 3.5 we have that the function $\beta_t(\cdot)$ is increasing and convex, and so is the drift function $a_t(\cdot)$. We can then fully characterize the regimes of the debt-to-asset process using the fixed points of the drift function $a_t(x) = \lambda_{t+1}^{-1}(0)h_t^{-1}(0)e^{\beta_t(x)}e^{-\mu+\sigma^2}$, which incorporates all model parameters.

Proposition 4.3. *The debt provision has a mean-reverting regime if and only if for each t the drift function $a_t(\cdot)$ either has two fixed points $0 < x_t < \bar{x}_t < \lambda_t^{-1}(0)$ or a unique fixed point $0 < x_t < \lambda_t^{-1}(0)$, in which case we set by convention $\bar{x}_t := \lambda_t^{-1}(0)$. The stability measure is given by $\mathcal{N}(\frac{\sigma}{2} + \frac{1}{\sigma} \log \frac{\bar{x}_t}{x_t})$. Moreover, if $\bar{x}_t < X_t < \lambda_t^{-1}(0)$, then we have $\mathbb{E}[X_{t+1} | (Y_1, \dots, Y_t)] > X_t$.*

If the drift function does not have a fixed point, the debt-to-asset process is always explosive. If there exist x_t, \bar{x}_t as in the proposition, then the debt-to-asset process is a mean-reverting process, as long as it does not exceed \bar{x}_t . We give its stability measure:

$$\begin{aligned} \mathbb{P}[X_{t+1} < \bar{x}_t | X_t = x_t] &= \mathbb{P}\left[\frac{x_t}{Y_{t+1}e^{-\mu+\sigma^2}} < \bar{x}_t\right] \\ &= \mathbb{P}\left[Y_{t+1} > \frac{x_te^{\mu-\sigma^2}}{\bar{x}_t}\right] = \mathcal{N}\left(\frac{\sigma}{2} + \frac{1}{\sigma} \log \frac{\bar{x}_t}{x_t}\right). \end{aligned} \quad (28)$$

The larger the distance between the long-run level and the instability level (in relation to the asset volatility), the larger the stability measure of the debt when it is in the mean reverting regime. When the stability measure is low, the debt-to-asset behaves for all practical purposes as an explosive process. If there is no fixed point of the drift function, then the debt only has an explosive regime.

Figure 7 illustrates the different regimes of the debt-to-asset. If the debt-to-asset process is in the mean reverting regime, but below x_t , the firm will leverage as it has high equity. A firm whose debt-to-asset is in the region $[x_t, \bar{x}_t]$ successfully deleverages by selling assets and repaying withdrawing lenders. In the explosive regime, it is not possible (under typical returns) to return to the mean reverting level, and the debt-to-asset is instead moving away from \bar{x}_t towards the debt ceiling, at which point the firm defaults.

Around the long-run leverage level, a positive fundamental return decreases the debt-to-asset ratio. Then, lenders increase leverage and push back the ratio to its long-run leverage. This is achieved by a decrease of the marginal belief, i.e., more pessimists invest. After a sequence of

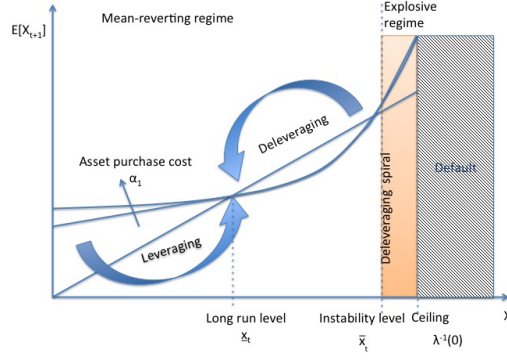


Figure 7: The expected future debt-to-asset as a function of the current debt-to-asset. Regimes of the debt-to-asset.

positive fundamental returns, the marginal belief is low. However, pessimists are the fastest to exit because their payoffs decrease the fastest under negative fundamental returns. If the debt-to-asset goes into the explosive regime because of negative fundamental returns, pessimists exit and would require higher and higher returns to reenter, which will not occur on a typical sample path of the exogenous fundamental returns. The explosive regime happens due to a spiral of withdrawals by pessimists.

We end this section by showing that the mean reverting level x_t is the level where the net inflow is zero (in expectation). Such characterization fully justifies calling this level “long-run level” (even if the long-run level is not a constant and may change in time). Suppose that $(1+r)e^{-\mu+\sigma^2} = 1$, i.e., $\mathbb{E}(\frac{1}{Y_t}) = \frac{1}{1+r}$: the expectation of the discount factor on the asset side is the same as the expectation of the discount factor on the debt side. Note that in this case, the asset grows faster than the debt ($e^\mu = (1+r)e^{\sigma^2} > (1+r)$). Intuitively, if the debt-to-asset process is mean reverting to the level x_t , then at this point the net debt inflow is zero, i.e. $D_t = D_{t-1}(1+r)$ if X_t were deterministic and equal to x_t ¹⁵. This writes $\Phi(\beta(x_t)) = x_t(1+r)$.

Proposition 4.4. *Suppose that $(1+r)e^{-\mu+\sigma^2} = 1$ and that there exists a fixed point $0 < x_t < \lambda_t^{-1}(0)$ such that $a_t(x_t) = x_t$. Then $\Phi(\beta(x_t)) = x_t(1+r)$, and $D_t = D_{t-1}(1+r)$ if $X_t = x_t$. Moreover, for $x > x_t$, $\beta(x)$ does not depend on the asset purchase cost α_1 .*

The proof of the Proposition 4.4 relies on the study of the dependence of the drift function on the asset purchase cost.

Remark 4.5. *The proposition states that the long-run leverage x_t is solution to the equation $\Phi(\beta(x_t)) = x_t(1+r)$. This is a direct computation that does not rely on the fixed point of the drift function $a_t(\cdot)$. It sets instead the much simpler condition of zero net inflow. Because x_t is solution to this equation and because $\beta(x)$ does not depend on the asset purchase costs for $x > x_t$, it follows that the long-run level x_t does not depend on the asset purchase costs.*

The important implication is that asset purchase costs can only change the speed of leveraging: the higher the asset purchase costs, the higher the speed to reach the long-run leverage level (the proof shows that $\alpha_1 \rightarrow a_t(\cdot)$ is increasing). But asset purchase costs do not change the long-run level and nor do they change the value of the drift function above the long-run level.

¹⁵Of course, since X_t is stochastic, on a typical path X_t will oscillate around x_t

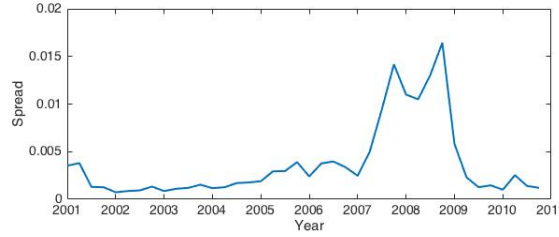


Figure 8: Spread (absolute value) between 3-Month AA Financial Commercial Paper and Effective Federal Funds Rate.

Above that level, we are in a deleveraging phase, so only the liquidation costs are relevant and not the asset purchase costs (see Figure 7).

5 Model dynamics under real world returns

We illustrate the dynamics of debt capacity in our model when the input is the series of real world asset returns. Notably, **we do not use any data on outstanding debt**. We use both a variable interest rate given by the actual spreads between short term debt rates and t-bill rates and a constant interest rate, given by the average spread over the period 2001 – 2005.¹⁶

We do not single out a firm, but we use aggregate data. We use the US Securities Industry Financial Results available at <http://www.sifma.org/research/statistics.aspx> for 40 quarters, from 2001 to 2010. The data contains “Aggregated income statement, selected balance sheet, and employment data on the U.S. domestic broker-dealer operations of all FINRA member firms doing a public business derived from their Financial and Operational Combined Uniform Single (FOCUS) Report filings”.¹⁷ Specifically, we let the fundamental returns $Y(0) = 1$ and $Y(t) = \frac{\text{Aggregate Revenue FINRA Members}_t}{\text{Aggregate Revenue FINRA Members}_{t-1}}$, $t = 1, \dots, 40$. We divide the data into two parts, and only use the first part, 2001 to 2005, to estimate the asset mean and the asset volatility. This is to ensure that model parameters are not calibrated using data during the financial crisis. We use the 3-Month Treasury Bill: Secondary Market Rate as a proxy for the risk-free rate (the outside option).¹⁸ We use the “3-Month AA Financial Commercial Paper Rate”¹⁹ to calibrate the interest rate offered to the lenders. The spread is plotted in Figure 8.

In summary, the calibrated model parameters are the following

- $T = 40$ periods, and $\delta = 1/4$ for the time length of one period.
- Asset return mean $\mu = 7\%$ and volatility $\sigma = 20\%$.
- The outside interest rate is set to the 3-Month Treasury Bill (quarterly data series 2001 – 2010, average spread 2001 – 2005 is 18 bp).

¹⁶We use the constant spread to insulate the model predictions from any information included in the actual spreads. Even under constant spread, the model predicts the collapse in debt capacity observed in the real world data.

¹⁷FINRA is responsible for regulating every US broker-dealer. The number of FINRA members in the time frame we consider averages 4000.

¹⁸Monthly data is available at <https://fred.stlouisfed.org/series/TB3MS>. We aggregate monthly data to obtain quarterly data. The outside option has been set to zero in the model exposition, but here we use the actual interest rate data to be consistent with using the returns of the asset as input.

¹⁹Monthly data is available at <https://fred.stlouisfed.org/series/DCPF3M>

- The interest rate is set to the 3-Month AA Financial Commercial Paper Rate.

The remaining parameters of the model could be calibrated if data was available. At this point, there is no data that would allow us to infer the belief distribution. We hope that our results would make a strong case for such data collection. We denote the variance of the belief distribution by σ_b . We let $\Phi(b) = (1 - \mathcal{N}((b - \mu)/\sigma_b))$.

Results are robust to variations in the parameters of this distribution, as we let $\sigma_b \in [0.15, 0.4]$. Likewise, we will vary the liquidation cost $\beta \in [0.08, 0.2]$ for our "aggregate" asset, but such costs can be estimated if the model was applied to a specific firm.

Figure 1a shows the real world dynamics of the debt, specifically the financial outstanding commercial paper, available at <https://fred.stlouisfed.org/series/FINCP>. This data is not used, but plotted there to compare against the dynamics resulting from our model. Figure 1b shows the dynamics of debt predicted by our model (the corresponding critical belief, debt-to-asset and the regimes of the debt-to-asset are shown in Figure 9).²⁰

In particular, as shown in Figure 1b our model predicts the collapse of the financial outstanding commercial paper in Q1 2008, just before the actual event in Q3 2008 after the fall of Lehman. The model predicts this collapse both when we use the real world spread series and when we use the constant spreads. The constant spread is the average spread over the period 2001 – 2005 and thus does not include any information about spreads in the crisis period. The collapse is therefore not driven by spreads, but by the debt-to-asset moving away from long-run level to the instability level.

The difference in debt under the actual spreads and under the average spread is minimal. Moreover, as shown in Figures 9a - 9b, the long-run level and the instability level given by the model change little when using the actual spreads vs. the average spread. As implied by our theory, debt collapses as leverage has a big deviation from the long-run level towards the instability level. The only difference from using the actual spreads is the more accurate calculation of the debt-to-asset X_t . Since spreads vary when the debt-to-asset was well above the mean-reverting level, it is not surprising that they have very little influence on the outcome of the deleveraging game: the pessimists' payoffs are too low at this level of leverage, and they are not very sensitive to the interest rate. Debt does not recover in our model after the collapse. Because of the convexity of their short put option, the pessimists would have required much higher positive returns to compensate the negative returns in order to stay with the firm.

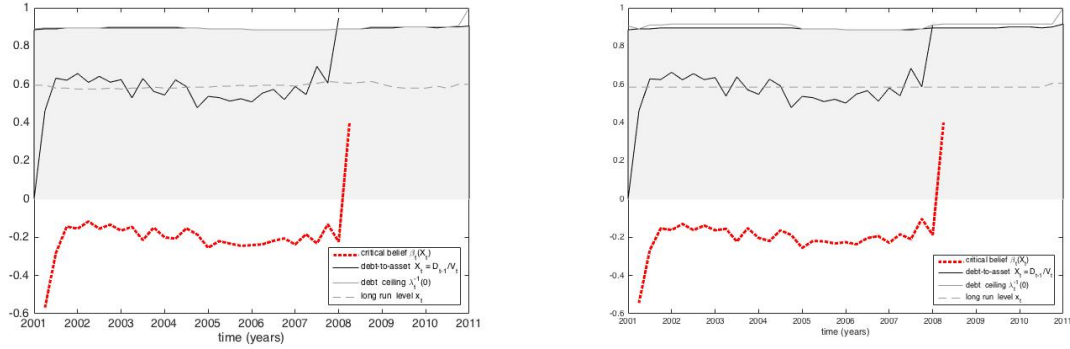
In the Appendix A.7, we demonstrate robustness of the model behavior under the real-world return data for a variety of parameters.

6 Comparative dynamics

In the following sections, we illustrate the dynamic behavior of our model under a variety of parameters and a variety of simulated paths of the fundamental trajectory. We assume that the beliefs are normally distributed around the true mean μ , with variance σ_b .

Model parameter values in the baseline case. We fix a time horizon of 10 years, and set $\delta = 1/4$ for the length of one period, i.e., $T = 40$. This means that lenders observe the firm's performance quarterly, and the short-term debt has is rolled over on a quarterly basis. We set $\sigma_b = 0.2$. The asset and money market parameters are given by an annualized volatility $\sigma = 6\%$, an annualized expected asset return $\mu = 3\%$, and an annualized short-term interest rate of $r = 1\%$, both in excess of the risk-free rate. The parameter values μ and σ for the log return distribution are

²⁰The parameters we use here are: $\sigma_b = 0.2$, $\beta = 0.1$ (liquidation cost).



(a) (Model) Debt-to-asset dynamics with real-world spreads. (b) (Model) Debt-to-asset dynamics with constant spread.

Figure 9: Debt collapses as leverage has a big deviation from the long-run level towards the instability level.

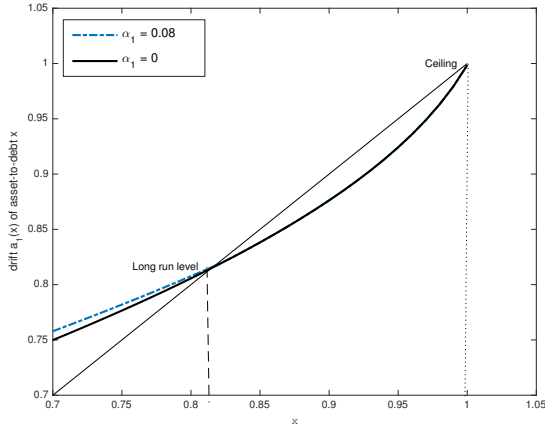
typical values for a diversified bond asset. If the firm is a large bank, the TED spread can be seen as a proxy for r . Its long term mean has been around 0.3%, but the spread has varied considerably over time, averaging between 1% and 2% during periods of financial distress such as in 2008 and 2009.

To make the analysis easy to follow, there is an overlapping case in each pair of consecutive figures, which show the drift function $a_t(x)$ of the debt-to-asset process for different parameters.

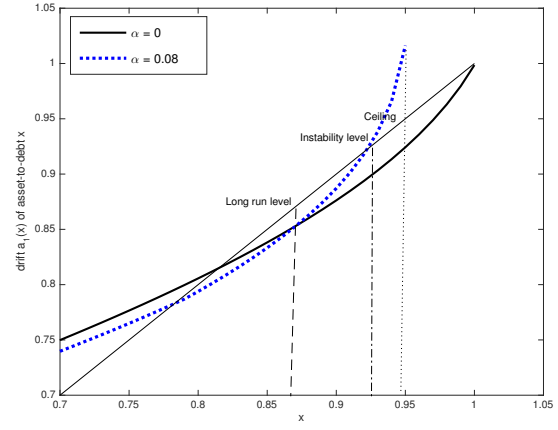
We start by analyzing the impact of the liquidation cost α and the asset purchase cost α_1 . We illustrate numerically Proposition 4.4 in Figure 10a. We set the liquidation cost $\alpha = 0$. We verify the points in Remark 4.5: 1) The drift function increases with the asset purchase cost (because the marginal belief function increases with the asset purchase cost); 2) The long run level (the first fixed point of the drift function) is the same for different asset purchase costs, and the drift functions coincide above this level. *Asset purchase costs have no impact on long-run levels of the leverage.* Of course, since asset purchase costs do lower the asset value, it means that debt levels are lower so that the long-run leverage is invariant with the purchase costs; 3) Below the first fixed point of the drift function, i.e., in the leveraging phase, the drift function increases with purchase costs. *Leveraging towards the long-run level is faster when purchase costs are higher.*

In the sequel, we set the asset purchase cost to zero. Figure 10b shows the effect of liquidation costs on the drift function. Unlike asset purchase costs, liquidation costs affect the long run level for the debt-to-asset ratio. One can see that: 1) The first fixed point of the drift function is lower if liquidation costs are higher. This means that *under higher liquidation costs, lenders impose a lower long-run leverage level*; 2) With zero liquidation costs, the ceiling is one. This means that *insolvency and default coincide under zero liquidation costs*: the firm defaults when the debt-to-asset reaches one (its equity is zero). In contrast, *when liquidation costs are higher, the ceiling is below one and the firm defaults while it is solvent* (its equity is non zero);

In the next two figures we analyze the role of the belief distribution and the exposure constraint. We set the liquidation cost $\alpha = 0.08$. In Figure 11a we investigate the effect of the variance of the belief distribution σ_b on the debt regimes. Larger heterogeneity of the beliefs translates in both a lower long run leverage and in a higher ceiling. The instability level is the same for all levels of belief heterogeneity. This means that the stability measure (given in (28)) increases with the belief heterogeneity as the ratio $\frac{\bar{x}_t}{x_t}$ between the instability level and the long run level is



(a) Liquidation cost α is zero.



(b) Asset purchase cost α_1 is zero.

Figure 10: The effect of the asset purchase cost (left) and the liquidation cost (right) on long-run level, instability level and ceiling. Levels are read where vertical lines intersect the diagonal.

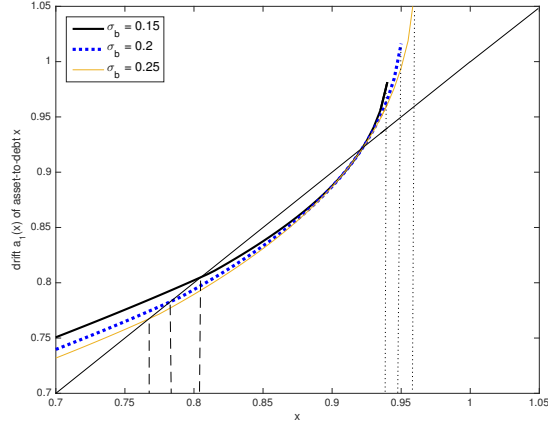
higher. When agents are less certain about the true mean of the asset returns, they impose a precautionary lower long-run leverage on the firm. At the same time the ceiling is higher so they allow the firm's debt-to-asset process to spend a longer time in the explosive regime which makes the early warning indicator more effective. *More belief heterogeneity makes debt more stable.* Under more heterogeneity, capital decreases slower as we go towards more optimistic beliefs.²¹ As we approach a point mass, the debt provision is most unstable: Assume that the marginal lender is slightly below the real world mean. Then a very small increase in the marginal lender is accompanied by a large loss of lender capital.

Next, in Figure 11b, we vary the maximum exposure constraint: lenders scale their maximum exposure to the borrower by γV_t . We set the variance of the belief distribution $\sigma_b = 0.15$. Recall that the debt capacity of one unit of the firm's asset is bounded from above by γ , see(2). Therefore γ is an upper limit on the long run leverage level: we verify that the long run leverage levels are below the respective values for γ .

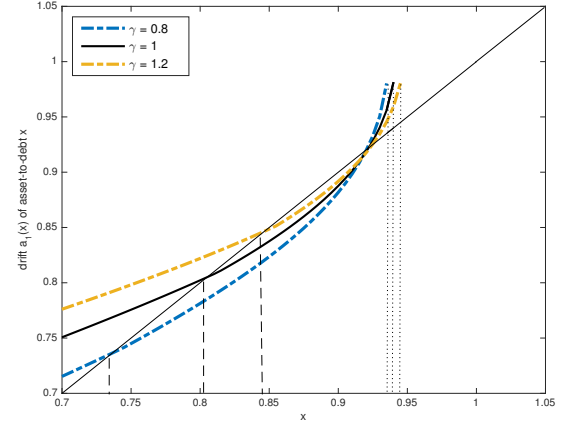
As expected, the ceiling increases with γ , as lenders are less constrained. However, it is not the ceiling that drives debt stability but the ratio between the instability level and the long run level, see (28). *More constraints on lenders' maximum exposure makes debt more stable.* Indeed, the instability level is the same for all γ considered, while the long run leverage level decreases as γ decreases. When the lenders are more constrained, they impose a precautionary lower long-run leverage on the firm. When lenders are less constrained, they allow for higher long-run leverage of the firm. *Debt is more unstable when lenders' constraints relax.*

Finally, we show that existence of an explosive regime depends on liquidation costs. In Figure 12, we show that under zero liquidation costs the debt-to-asset ratio is mean-reverting, and there is no explosive regime (both for $\gamma = 1.2 > 1$ and for $\gamma = 0.8 < 1$). Of course, the mean reverting regime is more stable in the sense of (28) when γ is smaller.

²¹ Around the mean ($|x - \mu| < \frac{\sigma_b}{\sqrt{2}}$) the gaussian density decreases with the variance σ_b .



(a) The effect of the belief variance σ_b .



(b) The effect of the exposure constraint γ . $\sigma_b = 0.15$.

Figure 11: The effect of the belief distribution and exposure constraint on long-run level, instability level and ceiling. Dashed vertical lines mark the long run average. Dotted vertical lines mark the ceiling.

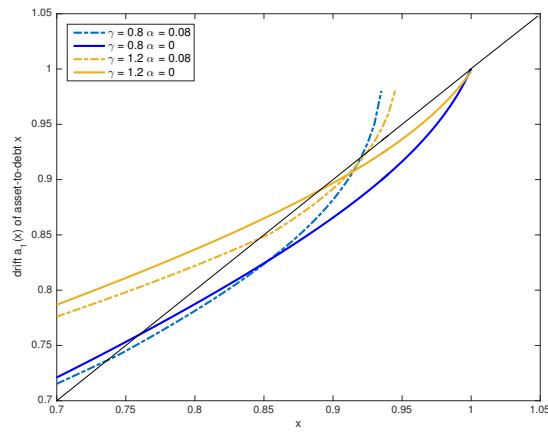


Figure 12: Effect of lenders' vs. effect of liquidation costs.

6.1 Debt dynamics

We now investigate the dynamics of the balance sheet. For the following simulations we set $\sigma_b = 0.2$, and $\alpha = 0.08$. For these parameters, the debt process has both a mean reverting and an explosive regime. We assume that the company starts with an initial asset value of $V_0 = 1$. The company's expected equity at the time horizon T under the (real-world) probability measure is given by $\mathbb{E}[(V_T - D_{T-1}(1+r))\mathbf{1}_{\{\tau > T\}}]$. We find this value by Monte Carlo simulation to be around 2.1. By raising short-term debt, the company therefore raises its annual expected rate of return on its initial equity from the fundamental asset's return of 3% to the rate of $\frac{1}{10} \log 2.1 = 7.4\%$. This increase in the expected return rate comes at the expense of introducing default risk; the Monte Carlo simulation yields a default probability under the real-world measure of around 0.01 over the 10-year period. The corresponding annualized default probability p satisfies $(1-p)^{10} = 1 - 0.01$, that is approximately $p = 0.1\%$. The expected value of one unit of the fundamental asset at the time horizon is 1.35, while the expected equity is 2.1. The Monte Carlo results indicate that only a small fraction of the paths end with default. In the vast majority of paths, for the considered parameters, the debt-to-asset process stays in the mean-reverting regime for the entire period.

In the figures below, we plot on the left-hand side the sample paths of the cumulative asset return (or the value of one unit of fundamental asset over time) $F_t = Y_1, \dots, Y_t$ along with the resulting debt process D_t , the equity process $V_t - D_{t-1}(1+r)$. On the right hand side, we plot the debt-to-asset process (which reflects the marginal belief, also plotted).

Figure 13 below shows a typical sample in which the debt-to-asset process stays in the mean-reverting regime for the entire period. Figure 14 below selects one of the paths with default, and shows that on that path the debt-to-asset switches from the mean-reverting regime to the explosive regime.

Steady state dynamics. For identification of the regimes of the debt-to-asset, we plot the long-run level x_t , the instability level \bar{x}_t and the ceiling. These quantities are deterministic, and of course coincide in Figures 13b and 14b. We require to be sufficiently far away from the time horizon to ensure that the system is in the “steady state” and that time to maturity is not a driver of the debt provision stability (the debt capacity is trivially equal to zero at maturity, because all debt is repaid at the horizon).

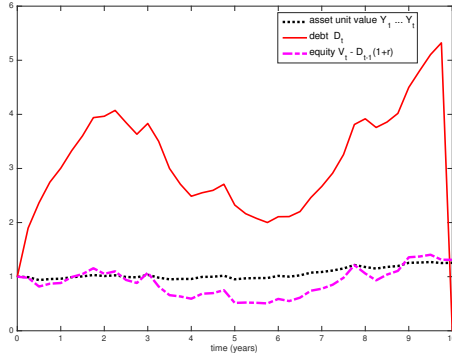
For time-independent spreads, risk free rate and belief distribution, the long-run level x_t , the instability level \bar{x}_t and the ceiling for the debt-to-asset ratio are also time-independent when the system is away from the time horizon. For the parameters we considered, the dynamics is in steady state for all but the last two (out of forty) periods.

Studying the stability of the debt-to-asset process necessitates sufficiently many periods to be far away from the time horizon. When we refer to the collapse in debt capacity, we refer to the collapse in debt capacity that occurs in “steady state” and not at maturity, as illustrated in Figure 14a.²²

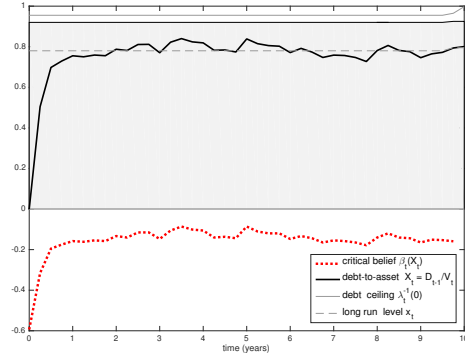
6.2 Hysteresis of the debt process

Zero liquidation costs. We set $\alpha = 0$ to insulate the model from the asymmetry in the transaction costs. We utilize the multi-period structure of our model to demonstrate the *path properties* of the

²²Our model cannot be built directly in steady state: a fixed time horizon is necessary because the arguments to eliminate the multiplicity of Nash equilibria are based on backward induction, as well as the non-trivial analysis of the monotonicity properties of the marginal lender's belief as a function of the debt-to-asset. In a companion working paper, available upon request, we prove convergence of marginal lender's belief functions as the time to the horizon increases. Achieving this convergence corresponds to the “steady state”.

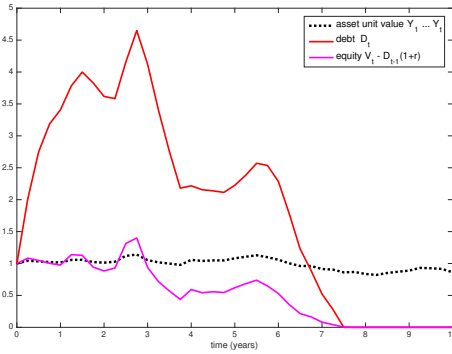


(a) A typical trajectory of the asset unit value and associated balance sheet dynamics

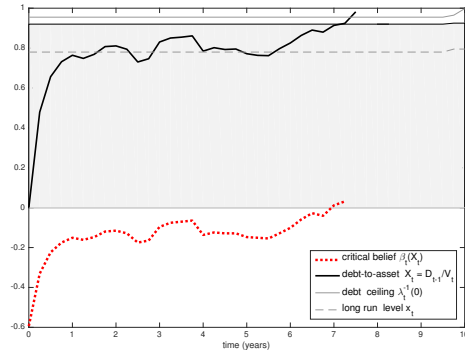


(b) The debt-to-asset ratio and the marginal belief exhibit a stationary behavior. In grey the mean reverting regime $(0, \bar{x}_t)$, with \bar{x}_t the instability level.

Figure 13: Debt dynamics with mean reverting leverage. The dynamics is in “steady state” (the long-run level, the instability level and the ceiling are time-independent) for all but the last two periods ($t \in [9.5, 10]$)



(a) A sample trajectory of asset unit value and associated balance sheet dynamics in which the firm defaults



(b) After a sequence of negative fundamental returns, the debt-to-asset ratio leaves reaches the instability level. After this the debt capacity collapses, ultimately leading to the firm's default. In grey the mean reverting regime $(0, \bar{x}_t)$, with \bar{x}_t the instability level.

Figure 14: Debt dynamics with default. The dynamics is in “steady state” (the long-run level, the instability level and the ceiling are time independent) for all but the last two periods ($t \in [9.5, 10]$). Default happens while in “steady state”, as the debt-to-asset switches from the mean reverting regime to the explosive regime as it crosses the instability level.

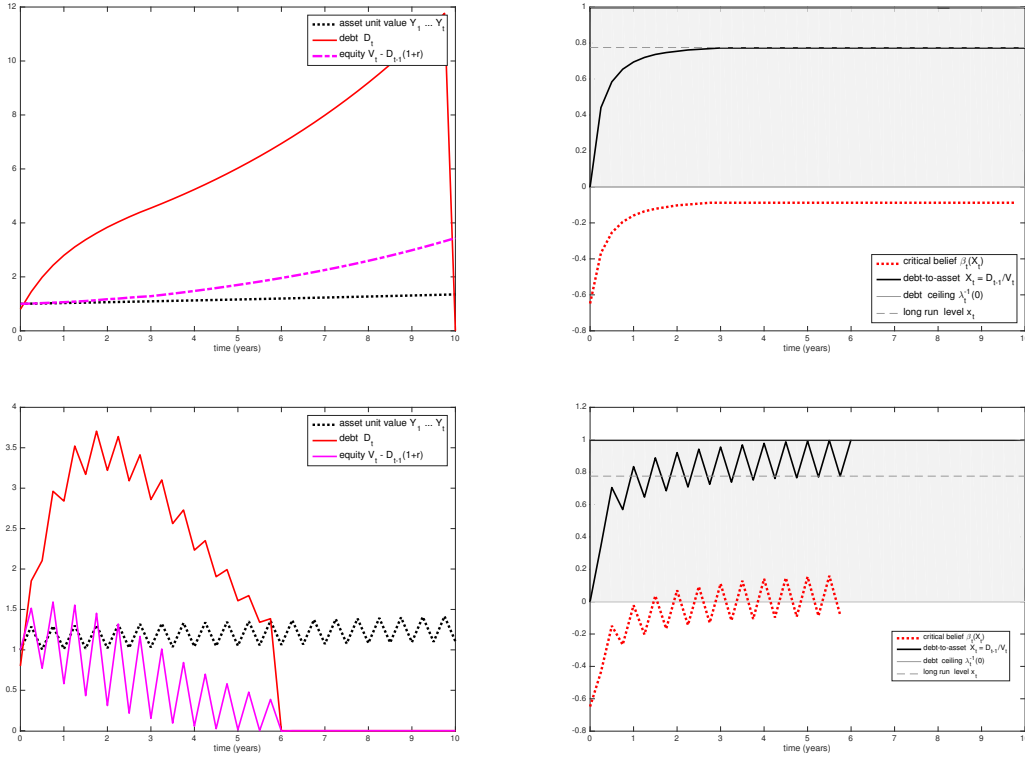


Figure 15: Debt process dynamics for zero liquidation cost ($\alpha = 0$). The diagrams show sample trajectories for the same model parameters with zero (above) and nonzero (below) realized volatility. The cumulative fundamental returns are the same at even times, but the realized volatility causes a downward drift in the debt process.

short-term debt dynamics. Debt presents a hysteresis phenomenon: paths of the fundamental return with different realized volatilities but with the same cumulative return lead to fundamentally different values for the debt.

To illustrate the dependence of the debt process on the realized volatility of the fundamental return process, we chose an annualized volatility of 20%, an annualized expected asset (excess) return of 3%, the exposure constraint $\gamma = 0.8$ and the standard deviation of the belief distribution $\sigma_b = 0.2$. In Figure 15, we compare the evolution of the debt process D_t for two trajectories with *the same realized fundamental return* F_T at maturity T (and in fact the same fundamental return in any even period), but with different realized volatility. The trajectory in the first example is given by $Y_t = e^R$ for all t , and in the second example it is given by $Y_t = e^{R+\sigma}$ for t odd and $Y_t = e^{R-\sigma}$ for t even. The results do not depend on the level of the fundamental value $F_t = Y_1 \cdots Y_t$; The qualitative behavior in the second case remains unchanged if we have $Y_t = e^{R-\sigma}$ for t odd and $Y_t = e^{R+\sigma}$ for t even.

In Figure 15 a downward trend in debt caused by the realized volatility and which is absent in the case of zero realized volatility. The problem of debt maturity choice has been studied in He and Milbradt (2016). Complementary to this work, our results indicate that a critical ingredient in the debt maturity choice is the response of the lenders' game to the realized volatility.

7 Bayesian updating of beliefs

So far we have treated the case when the belief distribution does not change over time. We now extend the analysis to a Bayesian setup. Each agent believes that the return Y_{t+1} is log-normal, with a *random* drift ν .²³ The conditional law of the log-return given ν is $\log Y_{t+1} | \nu \sim \mathcal{N}(\nu, \sigma^2)$. In the Bayesian setting, *the differentiated beliefs become differentiated priors*: An agent with belief b at time t has the prior $\nu \sim \mathcal{N}(b, \sigma_{prior,t}^2)$.

We are in the setting of conjugate priors: $\frac{1}{\sigma_{prior,t+1}^2} = \frac{1}{\sigma_{prior,t}^2} + \frac{1}{\sigma^2}$. Letting $\sigma_{prior,0} = \sigma_{prior}$, we have after t observations

$$\sigma_{prior,t}^2 = \frac{\sigma_{prior}^2}{1 + t \frac{\sigma_{prior}^2}{\sigma^2}}, \quad (29)$$

i.e., the variance of the agents' prior decreases to zero with time. The (inverse of the) parameter σ_{prior} gives the strength of the prior and drives how fast the agents learn. The larger σ_{prior} , the weaker the prior and the faster the updating of the agents' beliefs. In the case when $\sigma_{prior} \rightarrow 0$, the agents have strong priors, and their prior and posterior distributions coincide. This is the case of dogmatic beliefs pervasive in the literature.

We now determine the distribution of beliefs Φ_t . An agent with belief b_0 at time zero will have a posterior distribution after t observations

$$\nu \mid \log Y_1, \dots, \log Y_t \sim \mathcal{N} \left(\underbrace{\frac{t\sigma_{prior}^2}{\sigma^2 + t\sigma_{prior}^2} \bar{Y}_t + \frac{\sigma^2}{\sigma^2 + t\sigma_{prior}^2} b_0}_{\text{Belief at time } t}, \sigma_{prior,t}^2 \right), \quad (30)$$

with \bar{Y}_t the average of the observations, $\bar{Y}_t = \frac{\sum_{i=1}^t \log Y_i}{t}$. We now account for the heterogeneity in the beliefs. If $b_0 \sim \Phi_0 = \mathcal{N}(\mu, \sigma_b^2)$, then from (30) the distribution over the beliefs at time t is

$$\Phi_t = \mathcal{N} \left(\frac{t\sigma_{prior}^2}{\sigma^2 + t\sigma_{prior}^2} \bar{Y}_t + \frac{\sigma^2}{\sigma^2 + t\sigma_{prior}^2} \mu, \left(\frac{\sigma^2}{\sigma^2 + t\sigma_{prior}^2} \right)^2 \sigma_b^2 \right).$$

The standard deviation of the cross-distribution of beliefs Φ_t decreases to zero at speed $\frac{\sigma_{prior}^2}{\sigma^2}$. Thanks to the law of large numbers, $\bar{Y}_t \rightarrow \mu$ and the distribution Φ_t converges to a point mass around the true mean of the log-return μ . Of course the speed of convergence is controlled by σ_{prior} (and in extremis $\sigma_{prior} \rightarrow 0$ we are in the dogmatic case and $\Phi_t = \Phi_0$).

The mean of the belief distribution Φ_t is $\frac{t\sigma_{prior}^2}{\sigma^2 + t\sigma_{prior}^2} \bar{Y}_t + \frac{\sigma^2}{\sigma^2 + t\sigma_{prior}^2} \mu$. It depends on the observed log-return, but it converges to μ as the number of observations increases. The dependence on \bar{Y}_t complicates the algebra a lot. In particular one cannot express the expected return of an agent with belief b at time t as in proposition 3.3, since \bar{Y}_t will become an additional state variable.

Assumption 7.1. *We will assume that the real world drift per period μ is small compared to the variance of the belief distribution σ_b : $\mu \ll \sigma_b$.*

This assumption is verified when the size of the period is small, and in the numerical results we will take $\delta = 1/252$, i.e., daily frequency. Under this assumption, the fluctuation of the mean

²³To be precise, the drift must be adjusted by the volatility and is $\nu + \frac{\sigma^2}{2}$.

of Φ_t around the true mean μ is small compared to the decrease in the variance of the belief distribution. We then have the following approximation

$$\Phi_t \approx \mathcal{N}\left(\mu, \left(\frac{\sigma^2}{\sigma^2 + t\sigma_{prior}^2}\right)^2 \sigma_b^2\right). \quad (31)$$

For this approximate distribution in which the variance is time-dependent *the entire analysis goes through* as above, with minor modifications for the form of the expected payoff, see Appendix A.8.

We now analyze numerically debt stability under learning. The firm we have in mind is a fund or a dealer bank. The speed of learning is driven by the standard deviation of the prior σ^{prior} . We compare the case of dogmatic agents ($\sigma^{prior} = 0$) with the cases when $\sigma^{prior} = 0.05$ and $\sigma^{prior} = 0.1$ under different values of the exposure constraint γ . We consider here a shorter time horizon and a daily rollover and observation of the firm's asset: $T = 1$ and $\delta = 1/252$. The asset volatility is $\sigma = 0.25\sqrt{\delta}$, the average log-return is $\mu = 0.1\delta$. The standard deviation of the initial belief distribution is $\sigma_b = 0.25$ and thus we are under Assumption 7.1. The liquidation cost is $\alpha = 0.1$.

Learning makes debt more unstable. Figures 19, 20 and 21 show the debt dynamics for $\gamma = 1$ and varying $\sigma^{prior} \in \{0, 0.05, 0.1\}$. In all cases, the trajectory of the exogenous return Y_t is the same. In the dogmatic case, the long run level of the debt is lowest. Moreover, the instability level is lower than the debt ceiling. Thus, default has an early indicator.

When agents learn, $\sigma^{prior} = 0.05$, the long run level increases with time (and with the number of observations). When agents learn, they allow the firm to have a higher long run leverage. Importantly, the instability level becomes indistinguishable from the debt ceiling, meaning that default does not have any early indicators.

These effects are exacerbated in the case $\sigma^{prior} = 0.1$: the long run leverage increases faster with learning. As the long run level becomes closer to the debt ceiling, the stability measure for the debt decreases. This collapse in debt is in contrast to the case with no learning where the debt recovers. The same results hold for $\gamma = 0.9$ (Figures 22 - 23) and $\gamma = 1.1$ (Figures 24 - 25).

Learning increases the long-run level of the debt-to-asset. The continuum of agents behave in equilibrium as one risk-averse agent whose risk aversion is captured by the variance of the belief distribution. As this variance decreases by learning, the continuum of agents allows to firm to mean-revert around higher and higher levels of leverage. Because the distance between the long-run level and the instability level decreases, the measure of debt stability decreases (see (28)).

At the same time, learning makes any early indicators of default disappear. The intuition is clear: if all agents had the same belief, there could not be an early indicator of default. Indeed, as soon as the firm would cross the instability level, there would be an agreement that debt is explosive and is expected to reach the debt ceiling. Therefore, all lenders would run immediately when the instability level is crossed. By definition then the instability level and the debt ceiling would coincide. With learning, as the variance of the agents' beliefs decreases, the instability level and the debt ceiling coincide after several observations. In the dogmatic case the instability level is distinct from the debt ceiling because the marginal lender has a different belief than the real world drift of the asset (if such a marginal lender can be found). The leverage is not explosive under the marginal lenders' belief (by Remark 4.2 a marginal lender has a constant expected leverage). The early warning indicator is possible because of the difference in the expectation of the marginal lender and the real world expectation of the future leverage.

So learning has two opposite effects: it makes lenders "smarter" in detecting when leverage

becomes explosive, but at the same time it makes them (in aggregate) less risk averse and they allow the firm a much higher long-run leverage level. Overall, learning makes the endogenous leverage process more unstable.

8 Conclusions

We have endogenized the short-term debt by showing uniqueness of a Nash equilibrium in leveraging/deleveraging games with heterogeneous lenders. We have shown that the debt-to-asset has a mean-reverting and an explosive regime that we fully determine.

We constructed a parsimonious dynamic model that quantifies short term debt stability. The characteristics of the firm's asset can be estimated from data. The same is true the frequency. Much work remains to be done in particular to retrieve from data the belief distribution, which is the critical driver of our results. Our work makes the case to collect additional data that would allow us to calibrate the of lender's belief distribution, which we have shown is a critical driver of debt stability. In an empirical study, we have used as an input to the model the actual fundamental returns of the aggregate assets of FINRA members (security dealers). Our model, without using any debt data, predicts the collapse of the short term debt just before the actual event in Q3 2008. This prediction is robust.

We have extended the model to the case when agents learn the distribution of the asset return. Learning increases the long run leverage, and at the same time makes debt more unstable. As agents learn the true mean of the asset return, the early indicator of default disappears.

We have shown that short-term debt has trajectory properties and it can develop negative drift even in absence of long-term volatility. This hysteresis phenomenon suggests to adjust the frequency of rollover to the volatility of the asset, so that the debt-to-asset process can mean-revert. Then short-term debt is stable and can be used effectively.

Having shown the existence of an instability level for the debt-to-asset ratio, perhaps the most important practical implication emerging out of our study is the timing for intervention strategies. Such strategies would involve changing the firm's portfolio in conjunction with a signal to influence the lenders' belief distribution. These strategies would need to be implemented as soon as leverage approaches the instability level.

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A Appendix

All proofs are given in this appendix.

A.1 Proof of Proposition 3.3.

We consider the expected return per dollar from holding the firm's debt at time t for the period $[t, t+1]$ on the set $\{\tau^\pi > t\}$, given in (10). Let $\mathcal{F}_t = \sigma(Y_1, \dots, Y_t)$.

Under the measure \mathbb{P}^b we have that $\log Y_{t+1} \sim \mathcal{N}(b - \frac{1}{2}\sigma^2, \sigma^2)$ is independent of \mathcal{F}_t . Therefore (9) and a change of measure yield

$$\begin{aligned} R_t^{\pi,b} &= r - \mathbb{E}^b \left[\mathbf{1}_{\{\tau^\pi = t+1\}} \left(1 + r - f_t \left(\frac{V_{t+1}^\pi}{D_t^\pi} \right) \right) \mid \mathcal{F}_t \right] \\ &= r - (1+r) \mathbb{E}^b [\mathbf{1}_{\{\tau^\pi = t+1\}} \mid \mathcal{F}_t] + f \left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1} \left((1+r) \frac{D_{t-1}^\pi}{D_t^\pi} - 1 \right) \right) \mathbb{E}^b [\mathbf{1}_{\{\tau^\pi = t+1\}} Y_{t+1} \mid \mathcal{F}_t] \\ &= r - (1+r) \mathbb{P}^b [\tau^\pi = t+1 \mid \mathcal{F}_t] + f \left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1} \left((1+r) \frac{D_{t-1}^\pi}{D_t^\pi} - 1 \right) \right) \mathbb{P}^{b+\sigma^2} [\tau^\pi = t+1 \mid \mathcal{F}_t] e^b, \end{aligned} \quad (32)$$

where the quantity $\frac{D_{t-1}^\pi}{D_t^\pi}$ is well defined by, see Remark 2.7. Recall that, from (19), the default event $\{\tau^\pi = t+1\}$ is equal to $\left\{ Y_{t+1} e^{-b} < k_t \left(b, \frac{D_t^\pi}{V_t^\pi}, \frac{D_{t-1}^\pi}{V_t^\pi} \right) \right\}$. Set $k_t = k_t \left(b, \frac{D_t^\pi}{V_t^\pi}, \frac{D_{t-1}^\pi}{V_t^\pi} \right)$. On $\{\tau^\pi > t\}$ we have

$$f \left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1} \left((1+r) \frac{D_{t-1}^\pi}{D_t^\pi} - 1 \right) \right) = f \left(\frac{1}{\lambda_{t+1}^{-1}(0) e^b k_t} \right) = \frac{1-\alpha}{\lambda_{t+1}^{-1}(0) e^b k_t},$$

since $k_t > 0$ on $\{\tau^\pi > t\}$ and $\lambda_{t+1}^{-1}(0) > 0$ (see below). Hence (32) reads

$$\begin{aligned} R_t^{\pi,b} &= r - (1+r) \mathcal{N}(-d_2(k_t)) + f \left(\frac{V_t^\pi}{D_t^\pi} - f_t^{-1} \left((1+r) \frac{D_{t-1}^\pi}{D_t^\pi} - 1 \right) \right) \mathcal{N}(-d_1(k_t)) e^b \\ &= r - (1+r) \mathcal{N}(-d_2(k_t)) + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0) k_t} \mathcal{N}(-d_1(k_t)) \\ &= h_t(k_t). \end{aligned}$$

Finally we compute, using that $\varphi(-d_1(K)) = \varphi(-d_2(K))K$, where φ denotes the gaussian kernel,

$$\begin{aligned} h'_t(K) &= -\frac{1-\alpha}{\lambda_{t+1}^{-1}(0) K^2} \mathcal{N}(-d_1(K)) + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0) K} \varphi(-d_1(K)) \frac{1}{K\sigma} \\ &\quad - (1+r) \varphi(-d_2(K)) \frac{1}{K\sigma} \\ &= -\frac{1-\alpha}{\lambda_{t+1}^{-1}(0) K^2} \mathcal{N}(-d_1(K)) - \left(1+r - \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)} \right) \varphi(-d_2(K)) \frac{1}{K\sigma}. \end{aligned} \quad (33)$$

Since $\beta_{t+1}(\cdot)$ is strictly increasing, λ_{t+1} and λ_{t+1}^{-1} are strictly decreasing. By directly plugging in (16), we have

$$\lambda_{t+1} \left(\frac{1-\alpha}{1+r} \right) \geq 0,$$

so we obtain

$$\frac{1-\alpha}{1+r} \leq \lambda_{t+1}^{-1}(0). \quad (34)$$

It follows from (33) that $h'_t(K) < 0$. The limits for $K \rightarrow 0$ and $K \rightarrow \infty$ follow easily from the behavior of \mathcal{N} . \blacksquare

A.2 Properties of the strike as a function of the marginal belief

Recall from Proposition 3.3 that $h_t(K)$ is strictly increasing in K^{-1} with $h_t(K) \rightarrow -1$ for $K^{-1} \rightarrow 0$ and $h_t(K) \rightarrow r$ for $K^{-1} \rightarrow \infty$. In the following, in order to find the solutions to $R_t^{\pi, \beta_t(X_t^\pi)} = 0$ we shall analyze the behavior of the function $k_t(b, \Phi(b), X_t^\pi)^{-1}$ of the marginal belief $b = \beta_t(X_t^\pi)$ for a given value of X_t^π .

We now introduce two functions. The first is an upper bound for the marginal belief, or equivalently a lower bound on the necessary debt, that guarantees that the firm survives at time t , provided that π is a cutoff strategy at time t . We find it by observing that (7) and (15) yield

$$\begin{aligned} \tau^\pi > t &\Leftrightarrow 1 \geq f_t^{-1}((1+r)X_t^\pi - \Phi(\beta_t(X_t^\pi))) \\ &\Leftrightarrow 1 \geq ((1+r)X_t^\pi - \Phi(\beta_t(X_t^\pi)))(1-\alpha)^{-1} \\ &\Leftrightarrow \beta_t(X_t^\pi) \leq \Phi^{-1}((1+r)X_t^\pi - (1-\alpha)) =: \bar{b}_t(X_t^\pi) \end{aligned} \quad (35)$$

with the convention $\Phi^{-1}(x) = \Phi^{-1}(0)$ for $x < 0$ and $\Phi^{-1}(x) = -\infty$ for $x \geq \Phi(-\infty)$. Hence

$$\bar{b}_t(X_t^\pi) = \begin{cases} \Phi^{-1}(0) & \text{if } X_t^\pi \in [0, \frac{1-\alpha}{1+r}) \\ \Phi^{-1}((1+r)X_t^\pi - (1-\alpha)) & \text{if } X_t^\pi \in [\frac{1-\alpha}{1+r}, \frac{1-\alpha+\Phi(-\infty)}{1+r}]. \end{cases}$$

If $X_t^\pi > \frac{1-\alpha+\Phi(-\infty)}{1+r}$, the company defaults under any marginal belief.

The second function is defined piecewise on the sets $[0, \frac{1-\alpha}{1+r})$ and

$$\begin{aligned} \mathcal{S}_1 &:= \{x \mid x \geq \frac{1-\alpha}{1+r} \text{ and } \frac{1}{\phi(\Phi^{-1}((1+r)x))} + \frac{1}{(1+r)x} > \frac{1}{1-\alpha}\}, \\ \mathcal{S}_2 &:= \{x \mid x \geq \frac{1-\alpha}{1+r} \text{ and } \frac{1}{1+\alpha_1} \leq \frac{1}{\phi(\Phi^{-1}((1+r)x))} + \frac{1}{(1+r)x} \leq \frac{1}{1-\alpha}\}, \\ \mathcal{S}_3 &:= \{x \mid x \geq \frac{1-\alpha}{1+r} \text{ and } \frac{1}{\phi(\Phi^{-1}((1+r)x))} + \frac{1}{(1+r)x} < \frac{1}{1+\alpha_1}\}. \end{aligned}$$

It can be expressed via the function $\Psi(b) = \frac{1+\frac{\phi(b)}{\Phi(b)}}{\Phi(b)}$ (which is increasing for $b \in (-\infty, \Phi^{-1}(0))$) as

$$\beta_t^{\max}(X_t^\pi) := \begin{cases} \Phi^{-1}(0) & \text{if } X_t^\pi \in [0, \frac{1-\alpha}{1+r}) \\ \Psi^{-1}(\frac{1}{(1+r)X_t^\pi - (1-\alpha)}) & \text{if } X_t^\pi \in \mathcal{S}_1 \\ \Phi^{-1}((1+r)X_t^\pi) & \text{if } X_t^\pi \in \mathcal{S}_2 \\ \Psi^{-1}(\frac{1}{(1+r)X_t^\pi - (1+\alpha_1)}) & \text{if } X_t^\pi \in \mathcal{S}_3 \end{cases}$$

and is shown below to maximize the return of a marginal lender.

The function $\beta_t^{\max}(\cdot)$ is decreasing on $[0, \infty)$. To see this, first note that $\beta_t^{\max}(\cdot)$ is decreasing inside each of the sets $[0, \frac{1-\alpha}{1+r})$, \mathcal{S}_1 , \mathcal{S}_2 , and \mathcal{S}_3 . The monotonicity then follows by checking that $\beta_t^{\max}(\cdot)$ is continuous at the contact point of $[0, \frac{1-\alpha}{1+r})$ and \mathcal{S}_1 , of \mathcal{S}_1 and \mathcal{S}_2 , and of \mathcal{S}_2 and \mathcal{S}_3 .

Proposition A.1. *Let π be a cutoff strategy at time $t+1$ for some $t < T$ which satisfies (18). For any $X_t^\pi < \frac{1-\alpha+\Phi(-\infty)}{1+r}$, the inverse strike function $b \mapsto k_t(b, \Phi(b), X_t^\pi)^{-1}$ is strictly increasing for $b < \beta_t^{\max}(X_t^\pi)$ and strictly decreasing for $b > \beta_t^{\max}(X_t^\pi)$. It satisfies*

$$\lim_{b \rightarrow -\infty} k_t(b, \Phi(b), X_t^\pi)^{-1} = 0, \quad (36)$$

$$\lim_{b \rightarrow \bar{b}_t(X_t^\pi)} k_t(b, \Phi(b), X_t^\pi)^{-1} = 0 \quad \text{if } X_t^\pi \in (\frac{1-\alpha}{1+r}, \frac{1-\alpha+\Phi(-\infty)}{1+r}), \quad (37)$$

and its maximal value is given by

$$m_t(X_t^\pi) := k_t\left(\beta_t^{\max}(X_t^\pi), \Phi(\beta_t^{\max}(X_t^\pi)), X_t^\pi\right)^{-1} \quad (38)$$

$$= \begin{cases} \infty & \text{if } X_t^\pi \in [0, \frac{1-\alpha}{1+r}) \\ \frac{e^{\beta_t^{\max}(X_t^\pi)}}{\Phi(\beta_t^{\max}(X_t^\pi))} \left(1 - \frac{1}{1-\alpha}((1+r)X_t - \Phi(\beta_t^{\max}(X_t^\pi)))\right) \lambda_{t+1}^{-1}(0) & \text{if } X_t^\pi \in \mathcal{S}_1 \\ \frac{e^{\beta_t^{\max}(X_t^\pi)}}{(1+r)X_t} \lambda_{t+1}^{-1}(0) & \text{if } X_t^\pi \in \mathcal{S}_2 \\ \frac{e^{\beta_t^{\max}(X_t^\pi)}}{\Phi(\beta_t^{\max}(X_t^\pi))} \left(1 - \frac{1}{1+\alpha_1}((1+r)X_t - \Phi(\beta_t^{\max}(X_t^\pi)))\right) \lambda_{t+1}^{-1}(0) & \text{if } X_t^\pi \in \mathcal{S}_3. \end{cases}$$

The maximal value function $m_t(X_t^\pi)$ is decreasing in X_t^π .

Proof of Proposition A.1. For simplicity of notation, we shall write $X_t = X_t^\pi$ in this proof. We shall write \bar{b}_t and β_t^{\max} for $\bar{b}_t(X_t)$ and $\beta_t^{\max}(X_t)$.

From the definition of k_t in Proposition 3.3, we find

$$k_t(b, \Phi(b), X_t)^{-1} = \left(1 - f_t^{-1}((1+r)X_t - \Phi(b))\right) \frac{e^{b\lambda_{t+1}^{-1}(0)}}{\Phi(b)}. \quad (39)$$

To prove the statement of the proposition, by (39) it suffices to discuss the function

$$g_t(b) := \left(1 - f_t^{-1}((1+r)X_t - \Phi(b))\right) \frac{e^b}{\Phi(b)}$$

$$= \frac{e^b}{\Phi(b)} \left(1 - \psi(b)((1+r)X_t - \Phi(b))\right)$$

where we defined $\psi(b) := (1 - \mathbf{1}_{\{b \geq \bar{b}_t\}}\alpha + \mathbf{1}_{\{b < \bar{b}_t\}}\alpha_1)^{-1}$ and $\tilde{b}_t := \Phi^{-1}((1+r)X_t)$. Note that $\tilde{b}_t < \bar{b}_t$. Clearly we have $g_t(b) \rightarrow 0$ for $b \rightarrow -\infty$, whence (36). To see (37), note $X_t > \frac{1-\alpha}{1+r}$ implies $\Phi(\bar{b}_t(X_t^\pi)) = (1+r)X_t - (1-\alpha) > 0$. Now (39) and the equivalence $k_t(b, \Phi(b), X_t)^{-1} \geq 0 \Leftrightarrow b \leq \bar{b}_t(X_t)$ imply (37) and $g_t(b) > 0$ for all $b < \bar{b}_t(X_t)$.

To obtain the maximizer of $g_t(b)$, we first compute

$$g'_t(b) = \frac{e^b}{\Phi(b)} \left(1 + \frac{\phi(b)}{\Phi(b)}\right) \left(1 - \psi(b)((1+r)X_t - \Phi(b))\right) - \frac{e^b}{\Phi(b)} \psi(b) \phi(b)$$

$$= \frac{e^b}{\Phi(b)} \left(\left(1 + \frac{\phi(b)}{\Phi(b)}\right) \left(1 - \psi(b)(1+r)X_t\right) + \psi(b)\Phi(b) \right).$$

In particular we find

$$g'_t(\tilde{b}_t-) = e^{\tilde{b}_t} \left(\left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)}\right) \left(\frac{1}{(1+r)X_t} - \frac{1}{1+\alpha_1}\right) + \frac{1}{1+\alpha_1} \right),$$

$$g'_t(\tilde{b}_t+) = e^{\tilde{b}_t} \left(\left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)}\right) \left(\frac{1}{(1+r)X_t} - \frac{1}{1-\alpha}\right) + \frac{1}{1-\alpha} \right)$$

and we note that

$$g'_t(\tilde{b}_t-) \geq g'_t(\tilde{b}_t+). \quad (40)$$

We now consider the following cases.

Case 1) $(1+r)X_t < 1-\alpha$.

In this case $\bar{b}_t = \Phi^{-1}(0)$, $g'_t(b) > 0$ for all $b \in (-\infty, \Phi^{-1}(0))$ and $g_t((-\infty, \Phi^{-1}(0))) = (0, \infty)$.

Case 2) $1 - \alpha \leq (1 + r)X_t$ and $g'_t(\tilde{b}_t+) > 0$. The latter inequality is equivalent to

$$\frac{1}{\Phi(\tilde{b}_t)} \left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)} \right) \left(1 - \frac{(1+r)X_t}{1-\alpha} \right) + \frac{1}{1-\alpha} = \frac{1}{\Phi(\tilde{b}_t)} \left(1 + \frac{\phi(\tilde{b}_t)}{(1+r)X_t} \right) - \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)} \frac{1}{1-\alpha} > 0$$

and hence to $\frac{1}{\phi(\Phi^{-1}((1+r)X_t))} + \frac{1}{(1+r)X_t} > \frac{1}{1-\alpha}$, that is $X_t \in \mathcal{S}_1$.

In this case $g'_t(\tilde{b}_t-) \geq g'_t(\tilde{b}_t+) > 0$. We then have for all $b < \tilde{b}_t$

$$g'_t(b) \frac{\Phi(b)}{e^b} = \left(1 + \frac{\phi(b)}{\Phi(b)} \right) \left(1 - \psi(b)(1+r)X_t \right) + \psi(b)\Phi(b)$$

which is greater than zero if $1 + \alpha_1 - (1+r)X_t > 0$, and if $1 + \alpha_1 - (1+r)X_t \leq 0$

$$\begin{aligned} g'_t(b) \frac{\Phi(b)}{e^b} &= \left(1 + \frac{\phi(b)}{\Phi(b)} \right) \left(1 - \psi(b)(1+r)X_t \right) + \psi(b)\Phi(b) \\ &\geq \left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)} \right) \left(1 - \psi(b)(1+r)X_t \right) + \psi(b)\Phi(\tilde{b}_t) = g'_t(\tilde{b}_t-) \frac{\Phi(\tilde{b}_t)}{e^{\tilde{b}_t}} > 0 \end{aligned}$$

because $\frac{\phi(b)}{\Phi(b)}$ is increasing. So $g_t(b)$ is increasing for $b < \tilde{b}_t$.

Moreover $g'_t(b) \frac{\Phi(b)}{e^b}$ is strictly decreasing for $b > \tilde{b}_t$. Since $g'_t(\tilde{b}_t+) > 0$ and $g'_t(b)$ is negative for $b \rightarrow \bar{b}_t$, the function $g_t(b)$ achieves a unique maximum at some $\beta_t^{max} \in (\tilde{b}_t, \bar{b}_t)$ and is increasing for $b < \beta_t^{max}$ and decreasing for $b > \beta_t^{max}$.

From $g'_t(\beta_t^{max}) = 0$ we obtain $\left(1 + \frac{\phi(\beta_t^{max})}{\Phi(\beta_t^{max})} \right) \left(1 - (1-\alpha)^{-1}(1+r)X_t \right) + (1-\alpha)^{-1}\Phi(\beta_t^{max}) = 0$ and hence

$$\frac{1 + \frac{\phi(\beta_t^{max})}{\Phi(\beta_t^{max})}}{\Phi(\beta_t^{max})} = \frac{1}{(1+r)X_t - (1-\alpha)}$$

and hence $\beta_t^{max} = \Psi^{-1}\left(\frac{1}{(1+r)X_t - (1-\alpha)}\right)$ and

$$g_t(\beta_t^{max}) = \frac{e^{\beta_t^{max}}}{\Phi(\beta_t^{max})} \left(1 - (1-\alpha)^{-1}((1+r)X_t - \Phi(\beta_t^{max})) \right).$$

Moreover, $g'_t(\beta_t^{max}) = 0$ implies

$$\frac{d}{dX_t} g_t(\beta_t^{max}) = \frac{\partial}{\partial X_t} g_t(\beta_t^{max}) + g'_t(\beta_t^{max}) \frac{d}{dX_t} \beta_t^{max} = \frac{\partial}{\partial X_t} g_t(\beta_t^{max}) < 0. \quad (41)$$

Case 3) $1 - \alpha \leq (1 + r)X_t$ and $g'_t(\tilde{b}_t-) \geq 0 \geq g'_t(\tilde{b}_t+)$. Similarly as in the previous case, we check that the latter inequality is equivalent to $\frac{1}{1+\alpha_1} \leq \frac{1}{\phi(\Phi^{-1}((1+r)X_t))} + \frac{1}{(1+r)X_t} \leq \frac{1}{1-\alpha}$, that is $X_t \in \mathcal{S}_2$.

We then have $g'_t(b) > 0$ for all $b < \tilde{b}_t$; this is clear if $(1+r)X_t \leq 1 + \alpha_1$, and if $(1+r)X_t > 1 + \alpha_1$ it follows from

$$\begin{aligned} g'_t(b) \frac{\Phi(b)}{e^b} &= \left(1 + \frac{\phi(b)}{\Phi(b)} \right) \left(1 - \psi(b)(1+r)X_t \right) + \psi(b)\Phi(b) \\ &> \left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)} \right) \left(1 - \psi(b)(1+r)X_t \right) + \psi(b)\Phi(\tilde{b}_t) = g'_t(\tilde{b}_t-) \frac{\Phi(\tilde{b}_t)}{e^{\tilde{b}_t}} \geq 0. \end{aligned}$$

Since $g'_t(\tilde{b}_t+) \leq 0$ and $g'_t(b) \frac{\Phi(b)}{e^b}$ is strictly decreasing for $b > \tilde{b}_t$, the function $g_t(b)$ is decreasing for $b > \tilde{b}_t$ and thus achieves its unique maximum at $\beta_t^{max} = \tilde{b}_t$. Also

$$\Phi(\beta_t^{max}) = (1+r)X_t.$$

and hence $\beta_t^{max} = \Phi^{-1}((1+r)X_t)$. We have $\beta_t^{max} < \bar{b}_t$, and

$$g_t(\beta_t^{max}) = \frac{e^{\beta_t^{max}}}{\Phi(\beta_t^{max})} = \frac{e^{\beta_t^{max}}}{(1+r)X_t}.$$

Case 4) $1 - \alpha \leq (1+r)X_t$ and $g'_t(\tilde{b}_t-) < 0$. Similarly as in the previous cases, we check that the latter inequality is equivalent to $\frac{1}{\phi(\Phi^{-1}(\frac{1}{(1+r)X_t}))} + \frac{1}{(1+r)X_t} < \frac{1}{1+\alpha_1}$, that is $X_t \in \mathcal{S}_3$.

In this case $g'_t(\tilde{b}_t+) \leq g'_t(\tilde{b}_t-) < 0$. We then have for all $b > \tilde{b}_t$

$$\begin{aligned} g'_t(b) \frac{\Phi(b)}{e^b} &= \left(1 + \frac{\phi(b)}{\Phi(b)}\right) \left(1 - \frac{1}{1-\alpha}(1+r)X_t\right) + \frac{1}{1-\alpha}\Phi(b) \\ &\leq \left(1 + \frac{\phi(\tilde{b}_t)}{\Phi(\tilde{b}_t)}\right) \left(1 - \frac{1}{1-\alpha}(1+r)X_t\right) + \frac{1}{1-\alpha}\Phi(\tilde{b}_t) = g'_t(\tilde{b}_t+) \frac{\Phi(\tilde{b}_t)}{e^{\tilde{b}_t}} < 0. \end{aligned}$$

So $g_t(b)$ is decreasing for $b > \tilde{b}_t$.

We thus have $\beta_t^{max} := \sup \{b \mid g'_t(b) \geq 0\} < \tilde{b}_t$. Also $\beta_t^{max} > -\infty$ since $g'_t(b)$ is increasing for b sufficiently small. We then have for all $b < \beta_t^{max}$

$$\begin{aligned} g'_t(b) \frac{\Phi(b)}{e^b} &= \left(1 + \frac{\phi(b)}{\Phi(b)}\right) \left(1 - \psi(b)(1+r)X_t\right) + \psi(b)\Phi(b) \\ &> \left(1 + \frac{\phi(\beta_t^{max})}{\Phi(\beta_t^{max})}\right) \left(1 - \psi(b)(1+r)X_t\right) + \psi(b)\Phi(\beta_t^{max}) = g'_t(\beta_t^{max}) \frac{\Phi(\beta_t^{max})}{e^{\beta_t^{max}}} = 0, \end{aligned}$$

so the unique maximum of $g_t(b)$ is achieved for some $\beta_t^{max} < \tilde{b}_t$.

From $g'_t(\beta_t^{max}) = 0$ we obtain $\left(1 + \frac{\phi(\beta_t^{max})}{\Phi(\beta_t^{max})}\right) \left(1 - \psi(b)(1+r)X_t\right) + \psi(b)\Phi(\beta_t^{max}) = 0$ and hence

$$\frac{1 + \frac{\phi(\beta_t^{max})}{\Phi(\beta_t^{max})}}{\Phi(\beta_t^{max})} = \frac{1}{(1+r)X_t - (1+\alpha_1)}$$

and hence $\beta_t^{max} = \Psi^{-1}\left(\frac{1}{(1+r)X_t - (1+\alpha_1)}\right)$ and

$$g_t(\beta_t^{max}) = \frac{e^{\beta_t^{max}}}{\Phi(\beta_t^{max})} \left(1 - (1+\alpha_1)^{-1}((1+r)X_t - \Phi(\beta_t^{max}))\right).$$

Moreover, $g'_t(\beta_t^{max}) = 0$ implies

$$\frac{d}{dX_t} g_t(\beta_t^{max}) = \frac{\partial}{\partial X_t} g_t(\beta_t^{max}) + g'_t(\beta_t^{max}) \frac{d}{dX_t} \beta_t^{max} = \frac{\partial}{\partial X_t} g_t(\beta_t^{max}) < 0. \quad (42)$$

Combining cases 1) to 4) proves the behavior of $b \mapsto k_t(b, \Phi(b), X_t)^{-1}$.

To show the monotonicity of the maximum value function $m_t(\cdot)$, first note that $m_t(\cdot)$ is decreasing on the sets \mathcal{S}_1 and \mathcal{S}_3 (by (41) and (42)) and on \mathcal{S}_2 (since $\beta_t^{max}(\cdot)$ is decreasing). The overall monotonicity of $m_t(\cdot)$ then follows from the continuity of $\beta_t^{max}(\cdot)$, and the resulting continuity of $m_t(\cdot)$, at the contact point of $[0, \frac{1-\alpha}{1+r})$ and \mathcal{S}_1 , of \mathcal{S}_1 and \mathcal{S}_2 , and of \mathcal{S}_2 and \mathcal{S}_3 . ■

Combining the behavior of the strike above and Proposition 3.3 we obtain the following corollary.

Corollary A.2. *The function $b \mapsto h_t(k_t(b, \Phi(b), X_t))$ is strictly increasing for $b < \beta_t^{max}(X_t^\pi)$ and strictly decreasing for $b > \beta_t^{max}(X_t^\pi)$. Moreover*

$$\begin{aligned} \lim_{b \rightarrow -\infty} h_t(k_t(b, \Phi(b), X_t^\pi)^{-1}) &= -1, \\ \lim_{b \rightarrow \bar{b}_t(X_t^\pi)} h_t(k_t(b, \Phi(b), X_t^\pi)^{-1}) &= -1 \quad \text{if } X_t^\pi \in \left(\frac{1-\alpha}{1+r}, \frac{1-\alpha+\Phi(-\infty)}{1+r}\right), \end{aligned}$$

In particular, equation (22)

$$h_t(k_t(\beta_t, \Phi(\beta_t), X_t)) = 0.$$

with no roots or one root $-\infty < \hat{\beta}_t < \infty = \bar{\beta}_t$ on $X_t^\pi \in [0, \frac{1-\alpha}{1+r}]$, and has two roots $-\infty < \hat{\beta}_t < \beta_t < \bar{\beta}_t$ on $X_t^\pi \in (\frac{1-\alpha}{1+r}, \frac{1-\alpha+\Phi(-\infty)}{1+r})$.

A.3 Proof of Proposition 3.5

We prove the statement by backward induction. At time $t = T$ the claim is trivial. Let now $t < T$. By the induction hypothesis, $\lambda_{t+1}^{-1}(0)$ is well defined. Let x such that (22) has a solution. By (39) we have from the definition of β_t

$$\frac{1}{\lambda_{t+1}^{-1}(0)h_t^{-1}(0)} = k_t(\beta_t(x), \Phi(\beta_t(x)), x)^{-1} \frac{1}{\lambda_{t+1}^{-1}(0)} =: g(\beta_t(x), x) \quad (43)$$

with

$$g(b, x) = \left(1 - f_t^{-1}((1+r)x - \Phi(b))\right) \frac{e^b}{\Phi(b)}$$

and since $\beta_t(x)$ is the smallest solution to (22), we have $\frac{\partial}{\partial b}g(\beta_t(x), x) > 0$ by the monotonicity property of the function $b \mapsto k_t(b, \Phi(b), X_t^\pi)^{-1}$, see Proposition A.1. Clearly we also have $\frac{\partial}{\partial x}g(b, x) < 0$. Taking $\frac{d}{dx}$ in (43), we obtain

$$0 = \frac{\partial}{\partial b}g(\beta_t(x), x) \frac{\partial}{\partial x}\beta_t(x) + \frac{\partial}{\partial x}g(\beta_t(x), x),$$

which implies $\frac{d}{dx}\beta_t(x) > 0$.

To show convexity, we set

$$\psi(b, x) := (1 - \mathbf{1}_{\{b \geq \Phi^{-1}((1+r)x)\}}\alpha + \mathbf{1}_{\{b < \Phi^{-1}((1+r)x)\}}\alpha_1)^{-1} \quad (44)$$

and compute

$$\begin{aligned} \frac{d}{dx}\beta_t(x) &= -\frac{\partial}{\partial x}g(\beta_t(x), x) / \frac{\partial}{\partial b}g(\beta_t(x), x) \\ &= \frac{(1+r) \frac{e^{\beta_t(x)}}{\Phi(\beta_t(x))} \psi(\beta_t(x), x)}{\frac{e^{\beta_t(x)}}{\Phi(\beta_t(x))} \left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (1 - \psi(\beta_t(x), x)(1+r)x) + \psi(\beta_t(x), x)\Phi(\beta_t(x))} \\ &= \frac{1+r}{\left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) + \Phi(\beta_t(x))}. \end{aligned}$$

We have to show that the denominator of the last expression is decreasing in x . This is clear if $\psi(\beta_t(x), x)^{-1} - (1+r)x \leq 0$ (since Φ is log-convex). If $\psi(\beta_t(x), x)^{-1} - (1+r)x > 0$, note that for

all x with $(1+r)x - \Phi(\beta_t(x)) \neq 0$

$$\begin{aligned}
& \frac{d}{dx} \left(\left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) + \Phi(\beta_t(x)) \right) \\
&= \frac{\partial}{\partial b} \left(\frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))} \right) \beta'_t(x) (\psi(\beta_t(x), x)^{-1} - (1+r)x) - \left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (1+r) - \phi(\beta_t(x)) \beta'_t(x) \\
&= (1+r) \frac{\frac{\partial}{\partial b} \left(\frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))} \right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) - \phi(\beta_t(x))}{\left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) + \Phi(\beta_t(x))} - \left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (1+r) \\
&= (1+r) \frac{\left(\frac{\partial}{\partial b} \left(\frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))} \right) - \left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right)^2 \right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) - \left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) \Phi(\beta_t(x)) - \phi(\beta_t(x))}{\left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) + \Phi(\beta_t(x))} \\
&= (1+r) \frac{\left(\frac{\phi'(\beta_t(x))}{\Phi(\beta_t(x))} - 1 - 2 \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))} \right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) - \left(1 + 2 \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) \Phi(\beta_t(x))}{\left(1 + \frac{\phi(\beta_t(x))}{\Phi(\beta_t(x))}\right) (\psi(\beta_t(x), x)^{-1} - (1+r)x) + \Phi(\beta_t(x))} < 0
\end{aligned}$$

by (23). ■

A.4 Proof of Theorem 3.6

Using that $R_t^{\pi, b}$ is monotone increasing in the belief b , we easily check the following lemma, which states that a stable coalition at time t will necessarily have a monotonicity property. For those paths in which the firm survives, if a member of the coalition invests, then the members of the coalition with higher beliefs will fully invest their available capital.

Lemma A.3. *Consider a strategy π and $(y_1, \dots, y_t) \in \Gamma_t(\pi)$. Then for any stable coalition $B \subseteq \mathcal{B}$ (in the sense of Definition 2.8) we have*

$$\pi_t(b, y_1, \dots, y_t) = 1, \quad \forall b \in B, \quad b \geq \inf\{b \mid \pi_t(b, y_1, \dots, y_t) > 0\}.$$

We now proceed to the proof of the theorem.

Step 1) We prove that $\pi^{(t)}$ is a Nash equilibrium at time t , by backward induction on $t = T, \dots, 0$. The assertion is clear for $t = T$. Let now $t < T$. Since $\pi^{(t)}$ is of the form $\pi^{(t+1)}$, it follows from the induction hypothesis that $\pi^{(t)}$ is a Nash equilibrium at time $t+1$. By the monotonicity of $b \rightarrow R_t^{\pi^{(t)}, b}$ and the definition of $\hat{\beta}_t(\cdot)$, we have

$$\begin{aligned}
R_t^{\pi^{(t)}, b} > 0 &\Leftrightarrow b > \hat{\beta}_t(x_t^{\pi^{(t)}}(y_1, \dots, y_t)) \\
R_t^{\pi^{(t)}, b} < 0 &\Leftrightarrow b < \hat{\beta}_t(x_t^{\pi^{(t)}}(y_1, \dots, y_t)).
\end{aligned}$$

Therefore $\pi_t^{(t)}(b, \cdot) = 1_{\{R_t^{\pi^{(t)}, b}(\cdot) > 0\}}$ and we automatically verify the second part of Definition 2.6.

Step 2) We now show that $\pi^{(t)}$ is strongly coalition proof at time t as in Definition 2.8. Let $B \subseteq \mathcal{B}$ and $\tilde{\pi}$ a strategy which satisfies $\tilde{\pi}_s(\cdot, \cdot) = \pi_s^{(t)}(\cdot, \cdot)$ for all $s \neq t$, and $\tilde{\pi}_t(b', \cdot) = \pi_t^{(t)}(b', \cdot)$ for all $b' \in \mathcal{B} \setminus B$, and is stable. Fix $(y_1, \dots, y_t) \in [0, \infty)^t$. Note that $x_t^{\pi^{(t)}}(y_1, \dots, y_t) = x_t^{\tilde{\pi}}(y_1, \dots, y_t) =: x_t$. We consider the following cases, which correspond to points a) and b) in Definition 2.8.

a) $(y_1, \dots, y_t) \notin \Gamma_t(\pi^{(t)})$. We need to show that $y_t \notin \Gamma_t(\tilde{\pi})$. Assume for a contradiction that $y_t \in \Gamma_t(\tilde{\pi})$. Then there must exist a lender with belief $b_{\min} \in B$ with $b_{\min} < \hat{\beta}_t(x_t)$ who invests (since otherwise the liquidity capacity $f_t(V_t^{\tilde{\pi}}) + D_t^{\tilde{\pi}} - (1+r)D_{t-1}^{\tilde{\pi}}$ under $\tilde{\pi}$ would be less than the liquidity capacity $f_t(V_t^{\pi^{(t)}}) + D_t^{\pi^{(t)}} - (1+r)D_{t-1}^{\pi^{(t)}}$ under $\pi^{(t)}$, which is negative by (7)). Since the firm

survives at time t under $\tilde{\pi}$, by Lemma A.3, it follows that under $\tilde{\pi}$ all lenders in B with belief $b \geq b_{min}$ fully invest. By (21), any lender $b \in B$ with $b < \hat{\beta}_t(x_t)$ has negative expected return $R_t^{\tilde{\pi},b}$. Hence any such lender has an incentive to withdraw from the coalition, and so the coalition is not stable in contradiction to our assumption. Hence $y_t \notin \Gamma_t(\tilde{\pi})$, and thus we showed that $\Gamma_t(\tilde{\pi}) \subseteq \Gamma_t(\pi^{(t)})$.

b) $(y_1, \dots, y_t) \in \Gamma_t(\tilde{\pi})$. Since the firm survives at time t under $\tilde{\pi}$, then by Lemma A.3 there exists $\tilde{\beta}_t$ such that under $\tilde{\pi}$ all lenders in B with belief $b \geq \tilde{\beta}_t$ fully invest. So we can consider two cases. If $\tilde{\beta}_t \leq \hat{\beta}_t(x_t)$, then all lenders in $[\tilde{\beta}_t, \hat{\beta}_t(x_t)]$ have negative expected return, so the coalition is not stable. Hence the marginal belief $\tilde{\beta}_t$ under $\tilde{\pi}$ is above $\hat{\beta}_t(x_t)$. Then all lenders in $[\hat{\beta}_t(x_t), \tilde{\beta}_t] \cap B$ have zero return for the period $[t, t+1]$ under $\tilde{\pi}$, but positive expected return for the period $[t, t+1]$ under $\pi^{(t)}$, and thus $R_t^{\pi^{(t)},b} \geq R_t^{\tilde{\pi},b}$. ■

A.5 Weakly dominated strategies

The intuition behind weakly dominated strategies is as follows. Consider a lender who chooses her strategy under uncertainty about the other lenders' strategy, with the only assumption that if the firm survives, then the other lenders' strategy is an equilibrium. Strategy B is said to be *weakly dominated by strategy A* if A gives the lender a strictly better payoff than strategy B, no matter what the other lenders' equilibrium turns out to be, and moreover, she is indifferent between strategy A and strategy B if the outcome of the game will lead to the default of the firm. Given that in games with a continuum of players no change of one player's strategy will prompt the others to change their strategy, this definition is natural and the more usual iterated elimination of dominated strategies is not necessary.

We first note that the payoff at time t of any lender is the same for those strategies which lead to the default of the firm. Indeed, for a new lender, in case of default she will receive back her investment in full, whatever this investment is, so her payoff is zero. Similarly, an old lender will receive the firm's proceeds in proportion to her old investment, so she is indifferent among those strategies leading to default at time t . It suffices to define weakly dominated strategies as those for which there exists a lender who can find another strategy which gives her a strictly better payoff in all equilibria *with survival of the firm*. The "weak" dominance refers to the indifference to those strategies in which the firm defaults.

Definition A.4 (Weak dominance). *Let $t < T$. We say that a strategy π is weakly dominated on a trajectory with default at time t , $(y_1, \dots, y_t) \in \Gamma_{t-1}(\pi) \setminus \Gamma_t(\pi)$ ²⁴ if there exists a belief b and a strategy ξ for this belief such that*

$$\xi R_t^{\pi^*,b} > \pi_t(b) R_t^{\pi^*,b}$$

for any Nash equilibrium π^ at time t with $(y_1, \dots, y_t) \in \Gamma_t(\pi^*)$.*

A.6 Proof of Theorem 3.7.

We prove the statement by backward recursion on t . Fix $\epsilon > 0$. Clearly the statement is true for $t = T$ with marginal belief function

$$\hat{\beta}_T(\cdot) \equiv \infty$$

since $\pi_T(b, \cdot) \equiv 0$ for all b .

²⁴This reads $(y_1, \dots, y_t) \notin \Gamma_t(\pi)$ and $(y_1, \dots, y_{t-1}) \in \Gamma_{t-1}(\pi)$.

For the induction step, let $t < T$, and assume that the statement is true for $t + 1$. Let π be a ϵ -coalition proof Nash equilibrium at time t in which agents do not use dominated strategies. By the induction hypothesis, we have that π is a cutoff strategy at time $t + 1$ with marginal belief function $\hat{\beta}_{t+1}(\cdot)$, and

$$\tau^\pi > t + 1 \Leftrightarrow \hat{\lambda}_{t+1}(X_{t+1}^\pi) \geq 0.$$

We will now show that π is a cutoff strategy at time t with marginal belief function $\hat{\beta}_t(\cdot)$

Step 1) Suppose that $(Y_1, \dots, Y_t) \in \Gamma_t(\pi)$, that is we are in the set $\{\tau^\pi > t\}$. Then by (11) we have

$$\pi_t(b, Y_1, \dots, Y_t) = \begin{cases} 1 & \text{if } R_t^{\pi, b} > 0, \\ 0 & \text{if } R_t^{\pi, b} < 0. \end{cases} \quad (45)$$

Moreover, the monotonicity of $b \rightarrow R_t^{\pi, b}$ implies that there exists a unique value b_t^* satisfying

$$\begin{aligned} R_t^{\pi, b} > 0 &\Leftrightarrow b > b_t^* \\ R_t^{\pi, b} < 0 &\Leftrightarrow b < b_t^* \end{aligned} \quad (46)$$

and by (20) (using Proposition 3.3 with $\lambda_{t+1}(\cdot) = \hat{\lambda}_{t+1}(\cdot)$ and the associated functions \hat{h}_t , \hat{k}_t and $\hat{\beta}_t$) we have

$$\hat{h}_t\left(\hat{k}_t\left(b_t^*, \frac{D_t^\pi}{V_t^\pi}, X_t^\pi\right)\right) = 0. \quad (47)$$

Combining (45) and (46) we obtain

$$D_t^\pi = V_t^\pi \Phi(b_t^*).$$

Plugging this into (47) yields

$$\hat{h}_t\left(\hat{k}_t\left(b_t^*, \Phi(b_t^*), X_t^\pi\right)\right) = 0. \quad (48)$$

By definition of $\hat{\beta}_t(\cdot)$, we also have

$$\hat{h}_t\left(\hat{k}_t\left(\hat{\beta}_t(X_t^\pi), \Phi(\hat{\beta}_t(X_t^\pi)), X_t^\pi\right)\right) = 0.$$

Suppose, for a contradiction, that the marginal belief b_t^* defined in (46) is not equal to the smallest solution $\hat{\beta}_t(X_t^\pi)$ of equation (48). Then by Corollary A.2, we have $b_t^* > \beta_t^{\max}(X_t^\pi)$, and

$$\begin{aligned} \hat{h}_t\left(\hat{k}_t(b, \Phi(b), X_t^\pi)\right) &> \hat{h}_t\left(\hat{k}_t(b_t^*, \Phi(b_t^*), X_t^\pi)\right) = \hat{h}_t\left(\hat{k}_t\left(\hat{\beta}_t(X_t^\pi), \Phi(\hat{\beta}_t(X_t^\pi)), X_t^\pi\right)\right) = 0, \\ \forall b \in B &:= (b_t^* - \epsilon, b_t^*) \cap (\hat{\beta}_t(X_t^\pi), b_t^*). \end{aligned}$$

It follows that if all lenders $b \in (b_t^* - \epsilon, b_t^*)$ choose to fully invest, then these lenders have a strictly positive expected return, since in this case the total debt is given by $V_t^\pi \Phi(b_t^* - \epsilon)$ and we have

$$\hat{h}_t\left(\hat{k}_t(b, \Phi(b_t^* - \epsilon), X_t^\pi)\right) > \hat{h}_t\left(\hat{k}_t(b_t^* - \epsilon, \Phi(b_t^* - \epsilon), X_t^\pi)\right) > 0, \quad \forall b \in B.$$

Note that the set B is non-empty whenever ϵ is sufficiently small.

Then compared to π , all lenders b in the coalition B are strictly better off by choosing to fully invest, given that all other lenders in $B \setminus B$ keep their investment strategy in the equilibrium π . Moreover, the above inequality shows that the ϵ -coalition is stable since no lender in B has an

incentive to unilaterally reduce the maximum investment. Therefore, π is not ϵ -coalition proof, in contradiction to our assumption.

This proves that $b_t^* = \hat{\beta}_t(X_t^\pi)$. Together with (45) and (46) we obtain that π is a cutoff strategy at time t with marginal belief function $\hat{\beta}_t(\cdot)$.

Step 2) It remains to show $\tau^\pi > t \Leftrightarrow \hat{\lambda}_t(X_t^\pi) \geq 0$ and for this it is sufficient to show " \Leftarrow ". So we assume that $\hat{\lambda}_t(X_t^\pi) \geq 0$ and we suppose for a contradiction, that $\tau^\pi = t$, i.e.,

$$(Y_1, \dots, Y_t) \notin \Gamma_t(\pi).$$

Then it follows that the liquidity capacity is negative, and thus by (8),

$$1 - \alpha \mathbf{1}_{\{t < T\}} + \frac{D_t^\pi}{V_t^\pi} - (1 + r)X_t^\pi < 0.$$

It follows that

$$\pi_t(b, Y_1, \dots, Y_t) < 1$$

for some $b > \bar{\beta}_t(X_t^\pi)$, where $\bar{\beta}_t(X_t^\pi)$ defined in (35) is the upper bound on the marginal belief which guarantees survival at time t (otherwise we would have $\frac{D_t^\pi}{V_t^\pi} \geq \Phi(\bar{\beta}_t(X_t^\pi))$ and thus $1 - \alpha \mathbf{1}_{\{t < T\}} + \frac{D_t^\pi}{V_t^\pi} - (1 + r)X_t^\pi \geq 1 - \alpha \mathbf{1}_{\{t < T\}} + \Phi(\bar{\beta}_t(X_t^\pi)) - (1 + r)X_t^\pi = 0$). Therefore the set

$$B := \{b > \bar{\beta}_t(X_t^\pi) \mid \pi_t(b, Y_1, \dots, Y_t) < 1\}$$

is non-empty.

We now check the conditions of Definition A.4 and show that strategy of any $b \in B$ is weakly dominated on the fundamental trajectory $(Y_1, \dots, Y_t) \notin \Gamma_t(\pi)$. By (11), any Nash equilibrium in which the firms survives is a cutoff strategy with an associated marginal belief function β_t . Corollary A.2 and $\hat{\lambda}_t(X_t^\pi) \geq 0$ then imply that $\beta_t \in [\hat{\beta}_t, \bar{\beta}_t]$.

The set of equilibria in which the firm survives is thus included in the set of cutoff equilibria π^* with marginal belief function at time t $\beta_t \in [\hat{\beta}_t, \bar{\beta}_t]$. A direct verification of Definition A.4 shows that any strategy of the lenders b in B is weakly dominated on the trajectory (Y_1, \dots, Y_t) by the following strategy

$$\xi = 1 \quad \forall b \in B.$$

This follows since for any $b \in B$ we have $b > \hat{\beta}_t$. Then, for any cutoff equilibrium π^* with marginal belief function at time t $\beta_t \in [\hat{\beta}_t(X_t), \bar{\beta}_t]$ (the set of marginal beliefs $[\hat{\beta}_t(X_t), \bar{\beta}_t]$ is a superset of the set of Nash equilibria in which firm survives) we have $R_t^{\pi^*, b} > R_t^{\pi^*, \beta_t} = 0$. Therefore $\xi R_t^{\pi^*, b} > \pi_t(b) R_t^{\pi^*, b}$, since in this case full investment maximizes the payoff under belief b . Then π has weakly dominated strategies in contradiction to our assumption. ■

Proof of Proposition 3.8.

We have $\tau > t$ if and only if

$$h_t(k_t(\beta_t, \Phi(\beta_t), X_t)) = 0. \tag{49}$$

has a solution $\beta_t < \infty$. The reverse implication follows from Corollary A.2 since in this case $\beta_t < \bar{\beta}_t$, which is the upper bound on the marginal belief that guarantees survival. Conversely, if

$\tau > t$, the marginal belief must satisfy (49) with $\beta_t < \infty$, using Remark 2.7. We obtain

$$\begin{aligned}
\tau > t &\Leftrightarrow (49) \text{ has a solution } \beta_t < \infty \\
&\Leftrightarrow h_t\left(k_t\left(\beta_t^{\max}(X_t), \Phi(\beta_t^{\max}(X_t)), X_t\right)\right) \geq 0 \\
&\Leftrightarrow h_t\left(\frac{1}{m_t(X_t)}\right) \geq 0 \\
&\Leftrightarrow \frac{1}{m_t(X_t)} \leq h_t^{-1}(0) \\
&\Leftrightarrow m_t(X_t) \geq \frac{1}{h_t^{-1}(0)}
\end{aligned}$$

where we used Proposition A.1 for the second equivalence. Hence we showed (25).

Proof of Proposition 4.4. We prove the statement by backward induction. We let the liquidation cost α and we vary the asset purchase $\alpha_1 > 0$. We let $\beta^{(\alpha_1)}$ the marginal belief function under the asset purchase cost α_1 , and $\lambda^{(\alpha_1)}$ the liquidity capacity function. From (43), the marginal belief function solves

$$\frac{1}{(\lambda_{t+1}^{(\alpha_1)})^{-1}(0)h_t^{-1}(0)} = g(\alpha_1, \beta_t^{(\alpha_1)}(x), x) \quad (50)$$

with

$$g(\alpha_1, b, x) = \left(1 - \psi(\alpha_1, (1+r)x - \Phi(b))((1+r)x - \Phi(b))\right)^{\frac{e^b}{\Phi(b)}},$$

see (44) for the definition of ψ . We now show by induction that $(\lambda_{t+1}^{(\alpha_1)})^{-1}(0)$ is independent of α_1 and that $\beta_t^{(\alpha_1)}(x)$ is independent of α_1 for $x > \underline{x}_t$. The claim is clearly true for $t = T$. We now let $t < T$ and assume that the claim is true for $t + 1$. Note that the function h_t is independent of α_1 , so the lefthand side of (50) is independent of α_1 .

For fixed b and x , $g(\alpha_1, b, x)$ is decreasing in α_1 . Since $\frac{\partial}{\partial b}g(\alpha_1, \beta_t^{(\alpha_1)}(x), x) > 0$, we have that $\alpha_1 \rightarrow \beta_t^{(\alpha_1)}(x)$ is increasing. Let now $x_t^{\alpha_1}$ such that $(1+r)x_t^{\alpha_1} = \Phi(\beta_t^{(\alpha_1)}(x_t^{\alpha_1}))$. Such a point exists and is unique because $\beta^{(\alpha_1)}$ is increasing and Φ is decreasing, and the limits are $\Phi(\beta_t^{(\alpha_1)}(0)) > 0$ and $\Phi(\beta_t^{(\alpha_1)}(\infty)) = 0$. Note now that for all $\alpha_1 > 0$, and $x > x_t^{\alpha_1}$ we have that $(1+r)x > \Phi(\beta_t^{(\alpha_1)}(x))$. Therefore, in a neighborhood of $\beta_t^{(\alpha_1)}(x)$, we have that $(1+r)x > \Phi(b)$, i.e., there are only liquidations, and not asset purchases. Thus the solution $\beta_t^{(\alpha_1)}(x)$ to (50) does not depend on α_1 for $x > x_t^{\alpha_1}$. Therefore, the fixed points $x_t^{\alpha_1}$ coincide for all α_1 , and we denote the common value x_t^0 : for all $\alpha_1 > 0$, $(1+r)x_t^0 = \Phi(\beta_t^{(\alpha_1)}(x_t^0))$, and $\beta_t^{(\alpha_1)}(x) = \beta_t^0(x)$ for $x > x_t^0$.

From (16) it is immediate to see that $\lambda_t^{\alpha_1}(x) > 0$ for $x < x_t^0$. Since $\beta_t^{(\alpha_1)}(x) = \beta_t^0(x)$, for $x > x_t^0$, it follows that $\lambda_t^{\alpha_1}(x) = \lambda_t^0(x)$ (the liquidity capacity function does not depend on the asset purchase cost for $x > x_t^0$), and so $(\lambda_t^{(\alpha_1)})^{-1}(0)$ is independent of α_1 . This completes the induction step.

It remains now to show that the point x_t^0 is the smallest fixed point of $a_t(x) = x$. We can write, from the first line of (26) that $a(x) = \frac{\Phi(\beta(x))}{1 - f_t^{-1}((1+r)x - \Phi(\beta(x)))}e^{-\mu+\sigma^2}$. We then have $a(x_t^0) = \Phi(\beta(x_t^0))e^{-\mu+\sigma^2} = (1+r)e^{-\mu+\sigma^2}x_t^0 = x_t^0$, under the assumption that $(1+r)e^{-\mu+\sigma^2} = 1$, so x_t^0 is a fixed point of the drift function.

Suppose that x_t^0 is not the smallest fixed point of the drift function $a^{\alpha_1}(\cdot)$. Then, the smallest fixed point depends on α_1 . Given (27) and the monotonicity of $\alpha_1 \rightarrow \beta^{\alpha_1}(x)$, it follows that $\alpha_1 \rightarrow a^{\alpha_1}(x)$ is increasing. So the smallest fixed point of the drift function, $\underline{x}_t^{\alpha_1}$, is increasing as a function

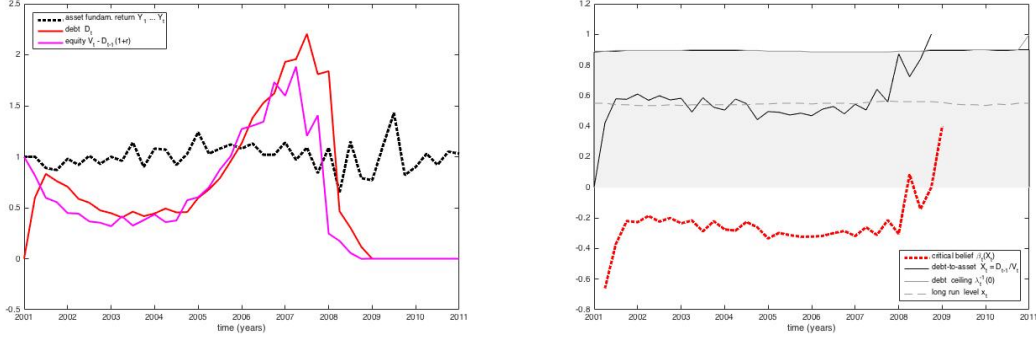


Figure 16: (Model) Debt dynamics under real-world returns. $\gamma = 0.6$, $\beta = 0.1$, $\sigma_b = 0.2$.

of α_1 . In words, the long-run level increases with the asset purchase costs. We now exploit the fact that $\underline{x}_t^{\alpha_1}$ is an long-run level to prove the contradiction.

We have that $\underline{x}_t^{\alpha_1} < x_t^0$ and let $X_t \in (\underline{x}_t^{\alpha_1}, x_t^0)$. At the same time, since x_t^0 is the first point where there are debt outflows, it follows that at X_t there are debt inflows. Under cost α_1 , the process will revert to the lower level $\underline{x}_t^{\alpha_1}$, so it will decrease.

Let $\epsilon = D_t - D_{t-1}(1+r) > 0$. Then the next period's asset position is $V_{t+1} = (V_t + \epsilon_1)Y_t$, with $\epsilon_1 = \frac{\epsilon'}{1+\alpha_1}$. Therefore, $\mathbb{E}[\frac{D_t}{V_{t+1}} - \frac{D_{t-1}}{V_t}] = \frac{\epsilon e^{-\mu+\sigma^2} V_t - \epsilon_1 D_{t-1}}{V_t(V_t + \epsilon)} < 0$, because the asset to debt ratio will revert to a smaller level than X_t . Therefore, we have that $X_t = \frac{D_{t-1}}{V_t} > (1 + \alpha_1)e^{-\mu+\sigma^2} = \frac{1+\alpha_1}{1+r}$.

Given that $X_t < x_t^0$, it follows that $\frac{1+\alpha_1}{1+r} < x_t^0$, and (because $(1+r)x < \Phi(\beta(x))$ for $x < x_t^0$) $1 + \alpha_1 = (1+r)\frac{1+\alpha_1}{1+r} < \Phi(\beta(\frac{1+\alpha_1}{1+r}))$. It now suffices to take $\alpha_1 \rightarrow \infty$ and use that the right hand side of the inequality decreases in α_1 to violate the inequality and obtain the desired contradiction. Therefore x_t^0 is the smallest fixed point of the drift function.

A.7 Robustness of predictions under real-world fundamental returns

In this section, we plot additional dynamics for the debt under the real-world return data, as we vary of parameters.

In these plots we set $\gamma = 0.6$. We note that the long-run level of leverage is lower than for the case $\gamma = 0.7$ in Section 5. Debt reaches a maximum of approximately 2. In unreported results, as we let $\gamma = 1$, the maximum debt and the long-run leverage increases. We also note that more heterogeneity of the belief distribution (captured by the standard deviation σ_b) makes the long-run level lower, and thus the maximum debt is lower in this case. The long-run level (and the maximum debt) is also lower when the liquidation costs increase. Despite variations in the actual level of debt and the long run leverage, all paths look qualitatively similar. In particular, the prediction of the debt collapse is robust.

A.8 Expected payoffs with learning

We let $\bar{k}_t = \bar{k}_t(\frac{D_t^\pi}{V_t^\pi}, \frac{D_{t-1}^\pi}{V_{t-1}^\pi}) = \frac{\frac{D_t^\pi}{V_t^\pi}}{\lambda_{t+1}^{-1}(0) \left(1 - f_t^{-1} \left((1+r) \frac{D_{t-1}^\pi}{V_{t-1}^\pi} - \frac{D_t^\pi}{V_t^\pi} \right) \right)}$ as in (19).

In the Bayesian setting, agent b believes that the real world drift of the asset return has a prior

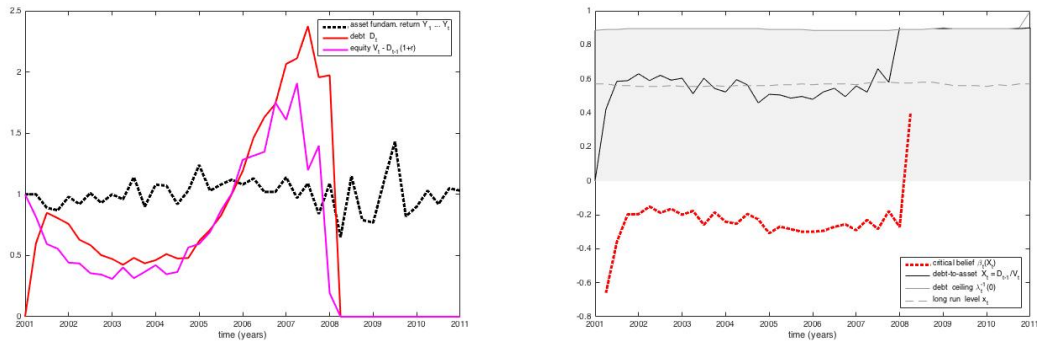


Figure 17: (Model) Debt dynamics under real-world returns. $\gamma = 0.6$, $\beta = 0.1$, $\sigma_b = 0.15$.

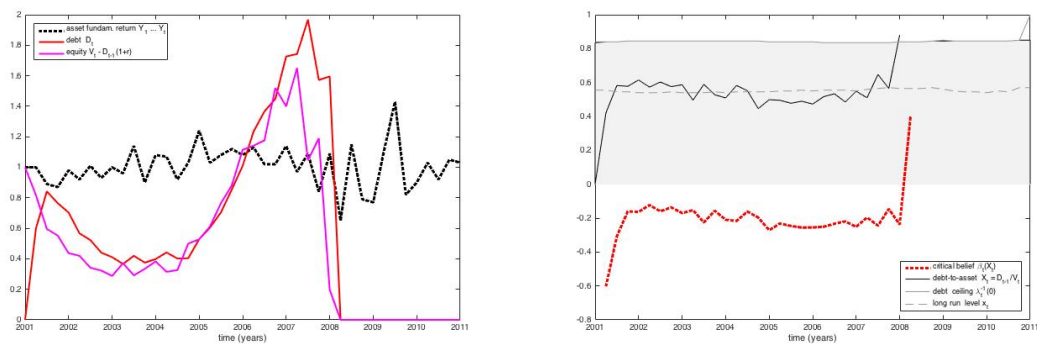


Figure 18: (Model) Debt dynamics under real-world returns. $\gamma = 0.6$, $\beta = 0.15$, $\sigma_b = 0.15$.

distribution centered around b . We thus integrate the payoff in (20) with respect to the prior:

$$\begin{aligned}
R_t^{\pi,b} &= \int_{\nu} \frac{1}{\sqrt{2\pi}\sigma_{prior,t}} e^{-\frac{(\nu-b)^2}{2\sigma_{prior,t}^2}} h_t(e^{-\nu}\bar{k}_t) d\nu \\
&= r - (1+r) \int_{\nu} \frac{1}{\sqrt{2\pi}\sigma_{prior,t}} e^{-\frac{(\nu-b)^2}{2\sigma_{prior,t}^2}} \mathcal{N}(-d_2(e^{-\nu}\bar{k}_t)) d\nu \\
&\quad + \int_{\nu} \frac{1}{\sqrt{2\pi}\sigma_{prior,t}} e^{-\frac{(\nu-b)^2}{2\sigma_{prior,t}^2}} \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)e^{-\nu}\bar{k}_t} \mathcal{N}(-d_1(e^{-\nu}\bar{k}_t)) d\nu \\
&= r - (1+r) \mathcal{N}\left(-d_2(e^{-b}\bar{k}_t)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right) \\
&\quad + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)\bar{k}_t} e^{b+\sigma_{prior,t}^2/2} \mathcal{N}\left(\left(-d_1(e^{-b}\bar{k}_t) - \frac{\sigma_{prior,t}^2}{\sigma}\right)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right) \\
&= r - (1+r) \mathcal{N}\left(-d_2(k_t)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right) \\
&\quad + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)k_t} e^{\sigma_{prior,t}^2/2} \mathcal{N}\left(\left(-d_1(k_t) - \frac{\sigma_{prior,t}^2}{\sigma}\right)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right). \tag{51}
\end{aligned}$$

We can write the expected return using

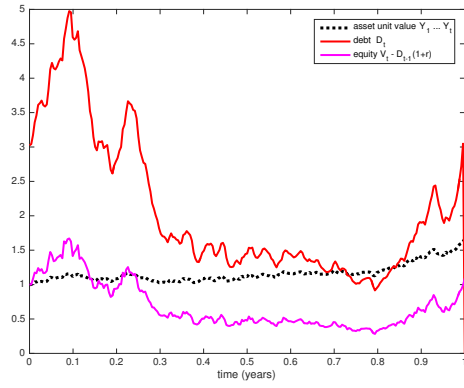
$$\begin{aligned}
h_t^{updated}(K) &:= r - (1+r) \mathcal{N}\left(-d_2(K)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right) \\
&\quad + \frac{1-\alpha}{\lambda_{t+1}^{-1}(0)k_t} e^{\sigma_{prior,t}^2/2} \mathcal{N}\left(\left(-d_1(K) - \frac{\sigma_{prior,t}^2}{\sigma}\right)/\sqrt{1+\frac{\sigma_{prior,t}^2}{\sigma^2}}\right). \tag{52}
\end{aligned}$$

In the particular case of dogmatic beliefs ($\sigma_{prior,t} = 0$) we recover the exact form of (20). The recursion in Definition 3.4 goes through (using $h_t^{updated}$) and the marginal belief β_t is obtained as the smallest solution to the equation

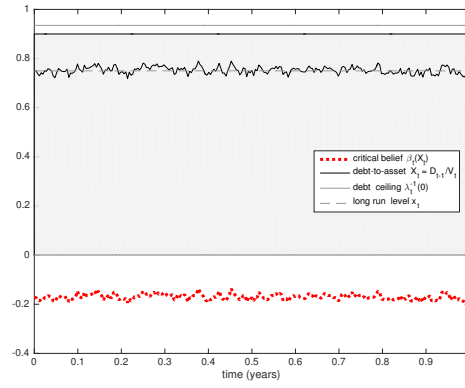
$$h_t^{updated}\left(k_t(\beta_t(X_t^\pi), \Phi_t(\beta_t(X_t^\pi)), X_t^\pi)\right) = 0, \tag{53}$$

where the time dependent belief distribution Φ_t is given in (31).

The marginal belief β_t corresponds to the unique equilibrium, as all proofs in Section 3.2 go through. The drift of the debt-to-asset is then given by (27). Proposition 4.3 gives the long-run level, the instability level as in the case without learning. We can now use computed levels for the leverage and demonstrate in Figures 19-25 the decrease in stability with the speed of learning.

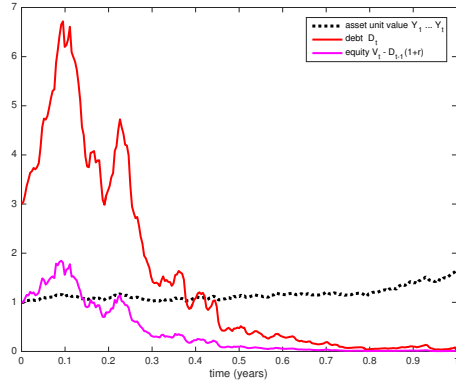


(a)

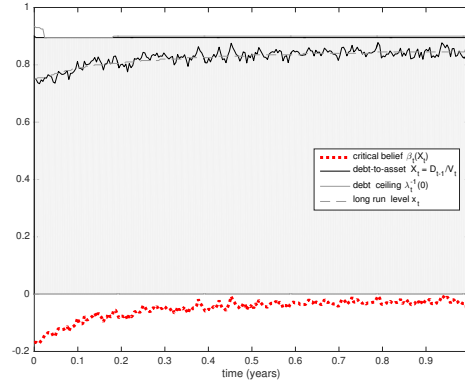


(b)

Figure 19: Debt and leverage dynamics: $\sigma_{prior} = 0$ (dogmatic), $\gamma = 1$.

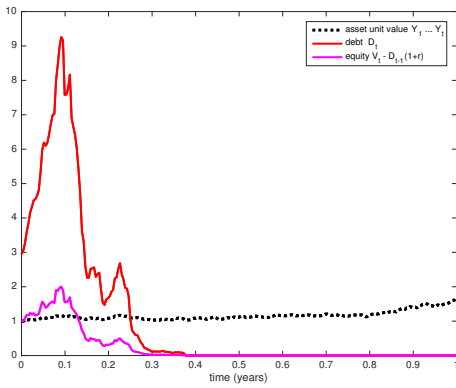


(a)

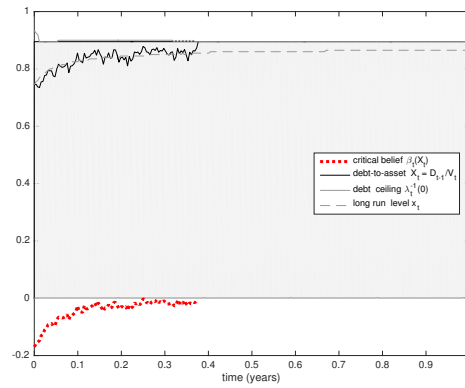


(b)

Figure 20: Debt and leverage dynamics: $\sigma_{prior} = 0.05$, $\gamma = 1$.

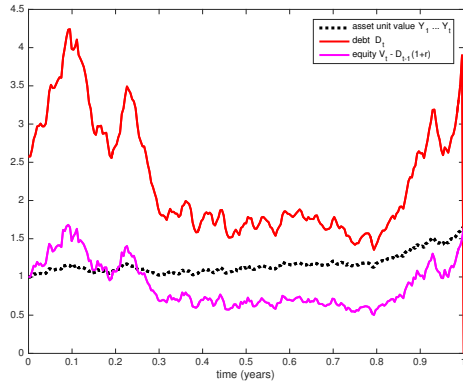


(a)

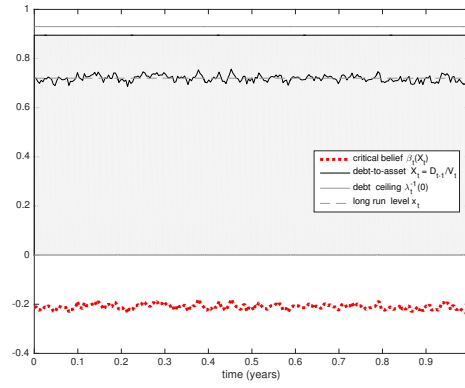


(b)

Figure 21: Debt and leverage dynamics: $\sigma_{prior} = 0.1$, $\gamma = 1$.

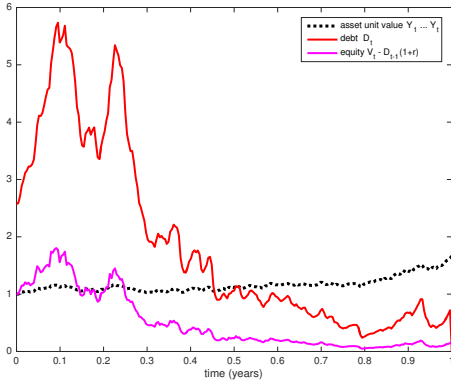


(a)

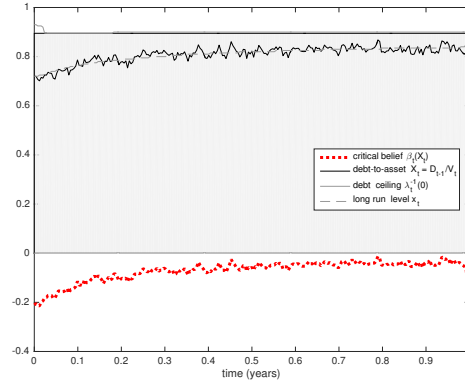


(b)

Figure 22: Debt and leverage dynamics: $\sigma_{prior} = 0$ (dogmatic case), $\gamma = 0.9$.

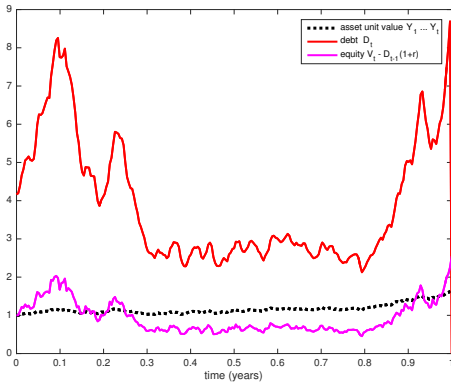


(a)

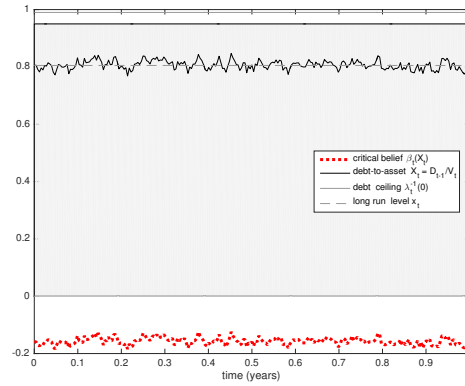


(b)

Figure 23: Debt and leverage dynamics: $\sigma_{prior} = 0.05$, $\gamma = 0.9$.

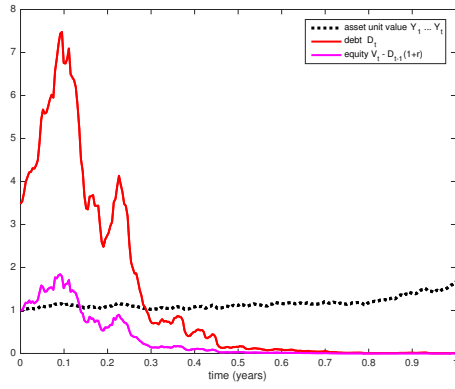


(a)

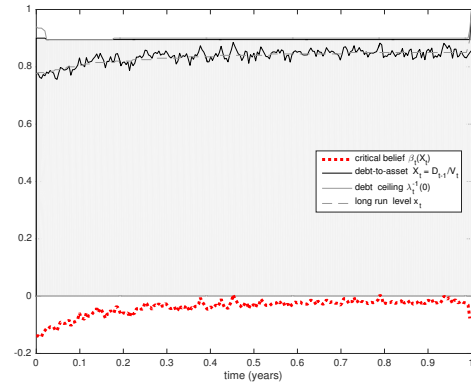


(b)

Figure 24: Debt and leverage dynamics: $\sigma_{prior} = 0$, $\gamma = 1.1$.



(a)



(b)

Figure 25: Debt and leverage dynamics: $\sigma_{prior} = 0.05$, $\gamma = 1.1$.