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"The Risk of Risk-Sharing: Diversification and Boom-Bust Cycles"

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The Risk of Risk-Sharing: Diversification and Boom-Bust Cycles*

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Abstract

In this paper, I model a shock whereby financial intermediaries can better diversify borrowers' idiosyncratic risks. A sector-specific diversification improvement induces intermediaries to reallocate funds toward the shocked sector. As lending spreads fall, intermediaries build up leverage over time. The result is a fragile sectoral boom that can end in an economy-wide bust. This cycle is amplified if the diversification-shocked sector is higher-risk or more external-finance dependent. I apply the model quantitatively to the recent housing cycle. Feeding in a novel mortgage diversification index, the model generates the measured increase in household credit coincident with a 1-2% decline in mortgage spreads. In the subsequent bust, spreads in all sectors spike by 2% as aggregate output drops.

JEL Codes: D14, G11, G12.

Keywords: Diversification, Financial Shocks, Credit Cycles, Housing Boom, Banking, Idiosyncratic Risk, Financial Innovation, Securitization, Deregulation.

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1 Introduction

Many booms feature a sectoral bias – for example, a US "railroad boom" in the 1850s, an "agriculture boom" in the 1880s, and a "housing boom" in the 2000s. Sectoral booms, however, can end in economy-wide busts. The aforementioned booms ended with the "Panic of 1857," the "Panic of 1893," and the "Great Recession." Securitization is an intriguing factor connecting these three episodes. In each case, loans to the booming sector were increasingly pooled into securities and sold to investors.[1] Motivated by these observations, this paper offers a financial theory that links sectoral booms to economy-wide busts.

I present a model in which financiers provide funds to two distinct productive sectors. A critical function of financiers' is to *diversify* within-sector idiosyncratic risks, which they accomplish by holding a large portfolio of loans. In practice, financiers diversify by making a variety of loans or holding securitized portfolios.

Suppose financiers' diversification improves within one of the sectors, as might occur with the advent of securitization. In the short run, a *reallocation effect* arises: facing an improved risk-reward trade-off, financiers redirect funds toward the sector with newly improved diversification, which raises sectoral investment. The reallocation effect helps explain why many booms feature a sectoral bias. Indeed, both the boom and its sectoral bias originate from sector-specific diversification improvements.

Meanwhile, better diversification reduces sectoral risk premia. A series of low risk premia earned by financiers results in a redistribution of wealth from financiers toward the rest of the economy. To maintain their funding activities, financiers must borrow more, which I call the *leverage effect*. If financial leverage is destabilizing, the leverage effect explains why a sectoral boom can lead to an economy-wide bust.

I adopt a particular connection between leverage and stability. Assume financiers face a leverage constraint. If the leverage effect is strong enough, financiers endogenously hit their constraint, at which point they must de-lever. A less-qualified type of financier, whom I call distressed investors, purchases financiers' liquidated loan portfolios and serves as the marginal supplier of any new loans. The de-leveraging thus disturbs both sectors, not only the sector that recently boomed. Lending spreads in both sectors rise sharply at the constraint, resembling a financial crisis.

A financial crisis might generate "real effects" on variables such as consumption and investment for several reasons. In the baseline model, I suppose the participation of distressed investors triggers deadweight losses. In reality, such losses might be justified by distressed investors' lower productivity in the advisory, monitoring, and screening activities that the financial sector typically provides. With deadweight losses, the de-leveraging episode triggers an inefficient bust. In the paper, I explore alternatives linking a financial crisis to an economy-wide bust.

[1] See Riddiough and Thompson (2012) and Calomiris and Schweikart (1991) for an account of the securitization of railroad-adjacent farm loans in the 1850s. See Eichengreen (1984), Snowden (1995), and Snowden (2007) for an account of farm mortgage securitizations in the 1880s. See below for evidence pertaining to the 2000s housing cycle.

Given this inefficiency, why do financial crises occur in equilibrium? The answer is a *risk-taking externality* in financiers' portfolio decisions. When individual financiers take risk, they do not account for the downward pressure they put on risk premia and, by extension, the profitability of other financiers. Lower financier profitability raises the prospect of future binding constraints, and hence a crisis. Although this prospect is socially undesirable, financiers privately ignore it.

The entire cycle is amplified if diversification improves in a *lower-quality* sector, i.e., a sector that is higher-risk or more reliant on external financing. Low-quality borrowers offload more risk onto financier balance sheets, and diversification has a larger marginal benefit when applied to a riskier balance sheet. This generates larger reallocation and leverage effects, meaning a larger and more asymmetric boom, but also a higher chance of a broad bust.

In a quantitative exercise, I apply the model to the recent US housing cycle. I create a novel index of idiosyncratic mortgage banking risk, in order to measure mortgage diversification. My approach has two key advantages relative to the existing empirical literature. First, the index encapsulates deregulation, financial innovations, and mergers, which tend to have similar qualitative effects but are difficult to compare quantitatively.[2] Second, the index has interpretable risk-based units, which is relevant for calibrating economic models. Using this index, I extract a time series measure of mortgage diversification, which increased substantially from 1990 to 2006; see figure 13.[3]

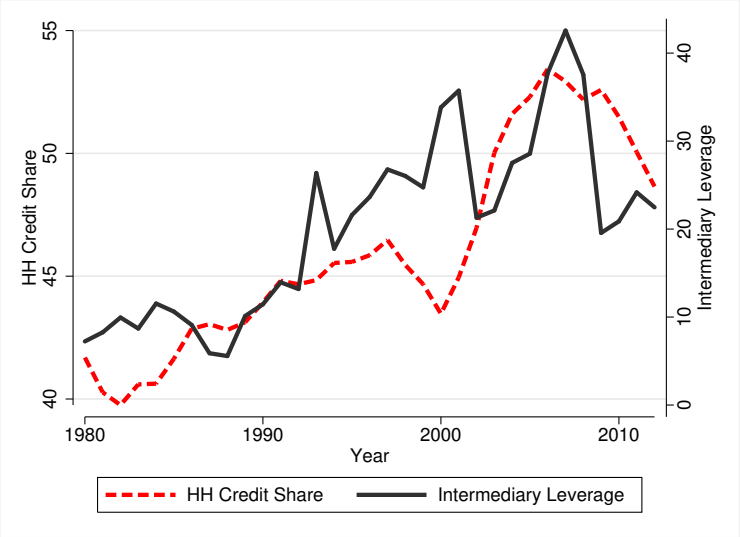

Figure 1: "HH Credit Share" denotes households' share of total non-financial corporate credit, from the Flow of Funds; "Intermediary Leverage" denotes broker-dealer leverage, from Adrian, Etula and Muir (2014).

Figure 1 shows the 1990-2006 increase in diversification is correlated with the reallocation and leverage effects. Reallocation is proxied by the household credit share, and leverage is proxied by broker-dealers' assets-to-equity ratio. Inserting my diversification time series into the calibrated

[2]For example, deregulations that allowed banks to operate across state borders clearly improved loan portfolio diversification. But the exact magnitude of this improvement is difficult to compare with the rise of securitization.

[3]I do not measure diversification in non-mortgage lending markets. But several facts suggest that deregulations and securitization were geared primarily towards household finance. First, the results of Rice and Strahan (2010) and Favara and Imbs (2015) together provide causal evidence that bank branching deregulations in the late 1990s and early 2000s disproportionately affected mortgage finance, relative to firm finance. Second, securitization of mortgages grew much faster in this period than securitization of commercial loans. The ratio of outstanding mortgage securities (primarily MBS) to corporate securities (primarily corporate bonds and CLOs) grew by 50% from 1990-2006. See Appendix D.2.

model, I match the household credit share in figure 1 and the 2% drop in mortgage rates documented in the literature.[4] Model-implied financier leverage also rises in the boom, qualitatively in line with figure 1. Because leverage constraints start binding in the bust, financiers' implied funding costs increase by over 2%, in line with data on financial crises.[5] The credit spreads of both sectors, not just housing, spike by the same magnitude of financiers' funding costs. In a counterfactual exercise without diversification improvements, there is no episode resembling a financial crisis, with spikes in funding costs or credit spreads.

While diversification improvements trigger a cycle characterized by sectoral reallocation and financier leveraging, other financial shocks might do the same. Motivated by the literature, I study five other financial shocks in the model – a loan-to-value shock,[6] a capital-requirement shock,[7] a risk-tolerance shock,[8] an uncertainty shock,[9] and a foreign-savings shock.[10] Among these, none generate both reallocation and leverage. The core rationale for this result is, unlike the other shocks, a diversification improvement differentially impacts one sector and differentially improves financiers' investment-opportunity set relative to other agents'. Although this analysis depends on the present framework, the forces uncovered are likely to be present in other model economies.

Contributions to the Literature

This paper contributes to three broad literatures: (1) the literature on the effects of financial intermediation on the macroeconomy; (2) the literature on diversification and other financial shocks; and (3) the literature on the recent housing cycle.

By focusing on the financial sector, my framework shares many features with the "financial accelerator" literature on macroeconomic dynamics with financial frictions. Net worth of borrowers, producers, or financiers acts as a buffer to fundamental economic shocks in these models, building off of insights by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). Like He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014), I employ continuous-time methods, to extend these ideas to study crisis dynamics and other nonlinearities.

The structure of my model and the structure common to this literature differ in three aspects. First, a key role of financiers in my model is to diversify idiosyncratic risks. Most of this literature studies financial intermediaries who are more productive investors but in fact less diversified than other agents. Second, I include two sectors, to study financiers' reallocation between them. Third, I study diversification improvements, a type of financial shock, which lead to interesting and differ-

[4] See Justiniano, Primiceri and Tambalotti (2017).

[5] See Fleckenstein and Longstaff (2018).

[6] LTV-type shocks are studied by Jermann and Quadrini (2012), Kiyotaki, Michaelides and Nikolov (2011), Justiniano, Primiceri and Tambalotti (2015b), Favilukis, Ludvigson and Van Nieuwerburgh (2017), and Greenwald (2016).

[7] This resembles the relaxation of banks' "lending constraints" in Justiniano, Primiceri and Tambalotti (2015a).

[8] For example, Kindleberger (1978) says, "The monetary history of the last four hundred years has been replete with financial crises. The pattern was that investor optimism increased as economies expanded, the rate of growth of credit increased and economic growth accelerated, and an increasing number of individuals began to invest for short-term capital gains..." See Kaplan, Mitman and Violante (2017) for an analysis of optimism shocks on housing markets.

[9] See Di Tella (2017) for a model of intermediation with idiosyncratic volatility shocks.

[10] See the "global savings glut" hypothesis of Bernanke (2005). Favilukis et al. (2017) study foreign savings in a model.

ent boom-bust dynamics. Most financial-accelerator papers focus on standard fundamental shocks, which are amplified by the endogenous concentration of risk in the financial sector. These models generate a bust only after a long sequence of negative fundamental shocks, because intermediaries are well-capitalized following a boom.[11]

By contrast, my economy can experience an endogenous bust which is then amplified by small negative fundamental shocks, because of the "leverage effect" and the presence of the financier leverage constraint. The leverage effect, whereby diversification lowers fundamental risk but is offset by higher risk-taking, relates to the "Peltzman effect" in automobile safety (Peltzman (1975)) or the "volatility paradox" in macro-finance (Brunnermeier and Sannikov (2014)).[12] The leverage constraint means that booms characterized by rising financial leverage can destabilize the economy, as asserted by Minsky (1992).[13]

The leverage constraint also has implications for efficiency. As financiers approach the constraint, less-qualified lenders are more likely to enter. To the extent that misallocation within the financial sector incurs real costs, this entry is inefficient. Without a leverage constraint, efficiency improves as financiers' risks fall (e.g., Brunnermeier and Sannikov (2014)).

The theoretical possibility that financial innovation may be inefficient is well-known. For example, Hart (1975) shows that partially "completing the market" by adding new securities markets can lead to lower welfare (see also Elul (1995)). In my paper, better financier diversification is a form of financial innovation that improves market completeness but can decrease welfare. The mechanism is through exacerbation of a pecuniary externality, whereby individual financiers do not internalize how their risk-taking decisions reduce the total equity of the financial system (see also Acharya, Le and Shin (2017)).

My approach to modeling diversification is related to the model in Gârleanu, Panageas and Yu (2015), which uses a Brownian bridge on a circle of locations to model correlated shocks. Investors allocate funds along arcs of the circle, which prevents full diversification of idiosyncratic risks. I use the theory of Gaussian processes to develop a new stochastic process, which I call a *Brownian cylinder*, that maintains cross-sectional correlations on a circle but accommodates an infinite-horizon, continuous-time setting. This apparatus could be useful in other settings where continuous-time methods have proved fruitful (e.g., optimal stopping problems, occasionally-binding portfolio con-

[11] This includes models like Di Tella (2017), in which the presence of uncertainty shocks endogenously induces risk concentration. Some models not based on net worth can generate busts following few adverse shocks. For example, Boissay, Collard and Smets (2016) present a model based on information asymmetries that generates a finance-centric boom-bust cycle. Gorton and Ordoñez (2016) generates cycles based on the interaction between real productivity and financiers' incentives to accept collateral.

[12] Demsetz and Strahan (1997) document this effect empirically for larger bank holding companies, whose better diversification is offset by increased risk-taking. Adrian, Friedman and Muir (2015) argue the negative correlation between financial sector ROE and expected returns supports the profitability mechanism in my paper. Also related, Wagner (2008, 2010, 2011) develops a series of theoretical models to illustrate downsides of financial diversification. Closest to my paper is Wagner (2008), in which the banking sector features a risk-taking externality. Individual banks do not take into account that high-risk, low-liquidity portfolio choices increase other banks' probability of inefficient liquidation. Better diversification improves risk-reward trade-offs, thereby worsening the externality.

[13] For example, Minsky (1992) says, "...Over periods of prolonged prosperity, the economy transits from financial relations that make for a stable system to financial relations that make for an unstable system."

straints, heterogeneous-agent macro models).

In my quantitative analysis, I apply the framework to the recent US housing boom. Motivating this exercise is a large empirical literature arguing credit-supply increases were the key driver of the boom.[14] For example, Favara and Imbs (2015) study the effect of credit on house prices using bank-branching deregulations of the late 1990s and early 2000s as a credit-supply instrument. The deregulations plausibly allowed banks to achieve better-diversified loan portfolios. My paper argues better mortgage diversification is an important credit-supply shock driving the boom and bust.

A quantitative modeling literature examines the plausibility of various financial shocks in the housing boom and bust. These papers are among the many papers (cited in the previous section) motivating my choice of "other financial shocks" to compare with diversification shocks. For example, Justiniano, Primiceri and Tambalotti (2015a) argue that a relaxation of lending constraints (a credit-supply shock) can generate a house-price boom, whereas a relaxation of borrowing constraints (a credit-demand shock) cannot, because of their opposing effects on real interest rates.

Regarding the bust, most of this quantitative literature incorporating financial shocks generates a bust only after applying a negative financial shock.[15] Missing from this literature is the idea that the nature of the boom can make the subsequent bust more likely and larger. Specifically, my model differs in that a diversification-induced boom creates financial instability.

Even under the narrative of a deterioration of household balance sheet (e.g., Mian and Sufi (2015)), the timing and extreme severity of the Great Recession are puzzling. Bernanke (2018) uses such arguments to conclude that the housing bust is not enough; the 2008 financial panic is key to a coherent narrative. My paper provides a model taking this finance view.

The remainder of the paper is organized as follows. Section 2 studies a baseline model to understand the reallocation and leverage effects from a diversification shock. Section 3 extends a baseline model to allow for the possibility of financial crises in response to the diversification shock. Section 4 shows how other financial shocks behave differently. Section 5 calibrates the model to quantify the effects of diversification improvements in the recent US housing cycle. Section 6 concludes. Appendix A contains model proofs. Appendix B details the construction and properties of the Brownian cylinder. Appendix C contains model extensions. Appendix D contains some empirical results.

[14] Adelino, Schoar and Severino (2012) use a regression-discontinuity design to show that (conforming) mortgage securitization reduces lending rates and raises house prices. Mian and Sufi (2018) provide causal evidence that the abrupt increase in private-label MBS activity dramatically increased house prices and led to the bust. See also Mian and Sufi (2009), Mian and Sufi (2011), and Di Maggio and Kermani (2017). This credit-supply view is not uncontroversial. For example, Haughwout, Lee, Tracy and Van der Klaauw (2011), Chinco and Mayer (2015), and Albanesi, De Giorgi and Nosal (2017) point to housing investors (who mortgage more than one property) as a key driver of the 2000s boom-bust cycle. Adelino, Schoar and Severino (2016) argue that mortgage credit increased proportionally to all income groups. These findings are at odds with the traditional rationing-based view, in which lower-quality borrowers are the largest beneficiaries of the credit-supply increase.

[15] For example, see Favilukis et al. (2017) and Kaplan et al. (2017). Guerrieri and Lorenzoni (2017) provide a thorough analysis of the effects of tightening borrowing constraints. Tightening constraints can generate a large bust and slow recovery, through a large deleveraging episode.

2 Two-Sector Model: Reallocation and Leverage

The model of this section is meant to introduce the primary mechanisms of my framework: reallocation and leverage. I introduce two sectors that produce with their own capital stocks. I will show that an increase in diversification of one sector's risks leads to reallocation towards that sector and, in the long run, to increased overall leverage by financial intermediaries.

2.1 Setup

Time is continuous $t \geq 0$. The model features two groups of agents: insiders and outside financiers. Insiders are additionally split into two groups, depending on which of two productive sectors they inhabit, A or B. These insiders invest in capital, and consume. To finance their capital purchases, insiders issue outside securities and put up some of their own net worth. These outside securities are held by financial intermediaries, which are operated by financiers. To finance their investment activities, financiers use their own net worth as well as risk-free debt. To start, financiers cannot directly manage productive capital. Agents in each group are indexed by $i \in [0, 1]$, which will represent an agent's location, to be described below. Figure 2 summarizes the model, with the flow of funds between insiders and financiers.

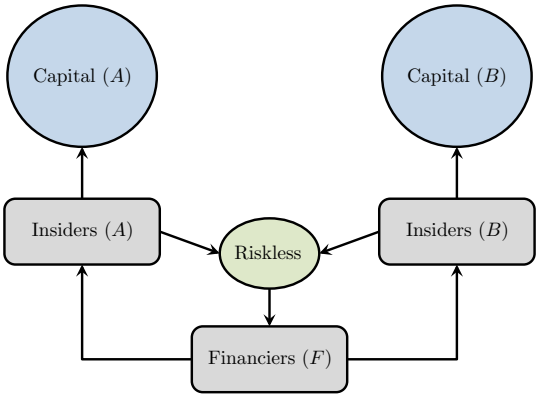

Figure 2: Typical Flow of Funds. Arrows show the direction of investment.

Preferences

All agents are infinitely-lived and have logarithmic utility over the single consumption good that may be produced in either sector. Mathematically,

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)} \log(c_s) ds\Big], \quad \rho > 0. \tag{1}$$

Locations and Idiosyncratic Risk

Agents are arranged on a circle, which has locations indexed by $i \in [0, 1]$. Locations will be special because they feature different idiosyncratic shocks.

These shocks directly hit the evolution of productive capital. Mathematically, capital held by an insider at location i evolves dynamically as

$$dk^A_{i,t} = dI^A_{i,t} + k^A_{i,t}[\sigma_A \cdot dZ_t + \hat{\sigma}_A dW^A_{i,t}] \tag{2}$$

$$dk^B_{i,t} = dI^B_{i,t} + k^B_{i,t}[\sigma_B \cdot dZ_t + \hat{\sigma}_B dW^B_{i,t}]. \tag{3}$$

In (2)-(3), dI^A, dI^B are desired investment, $Z := (Z^A, Z^B)$ is a standard Brownian motion (aggregate shock), and W^A, W^B are idiosyncratic shocks (more on these stochastic processes below). For simplicity, I assume $\sigma_A \cdot \sigma_B = 0$, i.e., orthogonal aggregate shocks. These "capital-quality shocks" are a simple way to capture permanent productivity or depreciation shocks, without introducing additional state variables.

For two reasons, I assume no investment adjustment costs, as in Cox, Ingersoll and Ross (1985). First, my focus is on incomplete financial markets rather than investment frictions. A minimal number of frictions affords maximum theoretical clarity, as my results on boom-bust cycles must be attributed to the remaining frictions.[16] Second, zero adjustment costs allows me to obtain analytical solutions to the equilibrium of this economy.

I assume the idiosyncratic shocks $W^A_{i,t}$ and $W^B_{i,t}$ are independent copies of a stochastic process with the following properties.

Assumption 1 (Shock Structure). *Assume the following for* $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$.

(i) At each location $i \in [0,1]$, $W_{i,t}$ *is a standard Brownian motion, independent of* Z_t.

(ii) For any two locations $i, j \in [0,1]$, *the shock correlation is*

$$\mathit{corr}(dW_{i,t}, dW_{j,t}) = 1 - 6\mathit{dist}(i,j)(1 - \mathit{dist}(i,j)), \tag{4}$$

where $\mathit{dist}(i,j) := \min(|i-j|, 1-|i-j|)$ *is a distance metric on the circle of circumference 1.*

(iii) $W_{i,t}$ *is continuous in* (i,t) *almost-surely, under the Euclidean distance metric on the cylinder* $\widetilde{\mathit{dist}}((i,s),(j,t)) := [|s-t|^2 + \mathit{dist}(i,j)^2]^{1/2}$.

Given part (i) of Assumption 1, $dW_{i,t}$ is iid over time, for fixed location i. Part (ii) of Assumption 1 means the shock correlations between locations decrease with their distance from one another.[17] Nearby locations have nearly perfect shock correlation. Two locations that are "far away" from one another (e.g., $i = 1/4$ and $j = 3/4$) will have a large negative correlation. A key question is whether any such stochastic process exists.

[16] By contrast, models such as Brunnermeier and Sannikov (2014) rely on capital adjustment costs to generate real effects of financial frictions. Indeed, in that model, the probability of capital misallocation vanishes as adjustment costs shrink to zero. Intuitively, the most productive users can avoid selling capital at a discount if they can disinvest costlessly.

[17] This presumption on the shock correlation owes to Gârleanu et al. (2015). Using the Brownian bridge on a "circle," they construct discrete-time idiosyncratic shocks that are cross-sectionally correlated but contain zero aggregate risk. In doing so, they find the dividend correlation is exactly $1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j))$. The proof of Lemma 2.1 would apply for any appropriate correlation function $v(i,j)$ that depends only on $\text{dist}(i,j)$ (i.e., stationary correlation function).

Lemma 2.1. *A stochastic process* $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$ *exists which satisfies Assumption 1.*

The preceding lemma, proved in Appendix B, establishes existence of W. The key step is proving W can be constructed as a Gaussian process with the appropriate covariance function, which needs to be symmetric and positive semi-definite. Because W evolves on a circle over time, which looks like a cylinder, I will call it the *Brownian cylinder*.

With the properties in Assumption 1, we can establish some distributional properties of the Brownian cylinder, in particular that it contributes no aggregate risk. These properties are stated below in Lemma 2.2.

Lemma 2.2. *Under Assumption 1, there is no aggregate risk, i.e.,* $\int_0^1 (dW_{i,t})di = 0$ *almost-surely. More generally, the local variance of a unit investment divided amongst the shocks along an arc of length* Δ *is equal to* $(1-\Delta)^2$*, i.e.,*[18]

$$Var_t\Big(\int_i^{i+\Delta} \Delta^{-1} dW_{j,t} dj\Big) = (1-\Delta)^2 dt.$$

Consequently, the process $W^{\Delta}_{i,t} := (1-\Delta)^{-1}\Delta^{-1}\int_i^{i+\Delta} W_{j,t} dj$ *is a standard Brownian motion.*

Given Lemma 2.2, the shock $dW_{i,t}$ is correlated across locations but washes out in the aggregate, the sense in which it is idiosyncratic. The surprising part of this result is that we only needed to specify the covariance structure of the shocks, and this property alone allows us to pin down the integral of all the shocks.

Figure 3 plots one simulation of the Brownian cylinder for $t \in [0,1]$ at 500 evenly spaced locations i. See Appendix B for details on how to simulate W. Each cross-section of the cylinder represents the circle of locations. To represent the shocks, the cylinder is shaded according to the size of $W_{i,t}/\sqrt{t}$.

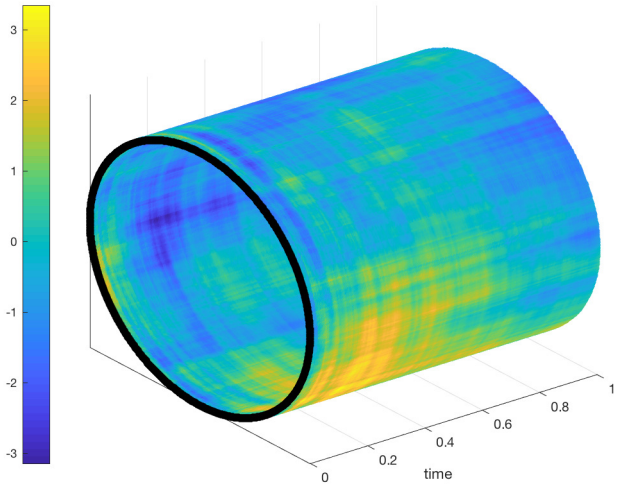

Figure 3: One shock realization of the Brownian cylinder $\{W_{i,t} : t \in [0,1]\}$, for i at 500 evenly spaced locations. Each cross-section of the cylinder is the circle of locations. Colors represent the size of $W_{i,t}/\sqrt{t}$.

[18] I always take the notational convention that "$i+\Delta$" represents $i+\Delta-\lfloor i+\Delta \rfloor$ when indexing a position on the circle.

Asset Markets

I assume sectoral capital is homogeneous, which implies the location-invariant unit prices $q_{A,t}$ and $q_{B,t}$. With zero investment adjustment costs, we will have $q_{A,t} \equiv q_{B,t} \equiv 1$ in equilibrium.

There is also a zero-net-supply futures market for trading claims directly on aggregate risk. Investing one unit of net worth in this claim earns the excess return $\pi_t dt + dZ_t$, where π_t is the market price of risk associated with the dZ_t shock. These futures contracts are continuously settled. Finally, there is a zero-net-supply riskless bond market that returns $r_t dt$. All agents can access both the futures and riskless bond markets frictionlessly.

Return on Capital

A firm is just a collection of capital, which produces according to an "AK" technology. The representative insiders at location i produce $G_A k^A_{i,t}$ and $G_B k^B_{i,t}$. As a result, and due to the absence of adjustment costs, the return on capital is given by

$$dR^A_{i,t} = G_A dt + \sigma_A \cdot dZ_t + \hat{\sigma}_A dW^A_{i,t} \tag{5}$$

$$dR^B_{i,t} = G_B dt + \sigma_B \cdot dZ_t + \hat{\sigma}_B dW^B_{i,t}. \tag{6}$$

Thus, capital returns have a location-invariant distribution and time-invariant risk.

Insider Problem

Because of symmetry between the two sectors and their insiders, I describe the problem of an insider in generic sector $z \in \{A, B\}$. On the asset side, insiders hold capital that returns (5)-(6). They are also marginal in the risk-free debt market, at the interest rate r_t. On the liability side, insiders can obtain funding from financiers against their capital, by signing a contract promising the return of

$$d\tilde{R}^z_{i,t} := (r_t + s^z_{i,t})dt + \left(dR^z_{i,t} - \mathbb{E}_t[dR^z_{i,t}]\right), \quad z \in \{A, B\}.$$

This liability is a way for insiders to shed some of the idiosyncratic risk associated with production. The "spread" charged by financial intermediaries is given by $s^z_{i,t}$.

I assume insiders finance a fixed fraction ϕ_z of the value of their enterprise from intermediaries in the form of outside equity. With the fixed fraction, insiders pay $\phi_z k^z_{i,t} d\tilde{R}^z_{i,t}$ to financiers. In Appendix A.4, such a risk-sharing arrangement is the (approximately) optimal solution to a standard moral-hazard problem.[19]

[19] Under that interpretation, the restriction that insiders keep $1 - \phi_z$ fraction of capital risk on their balance sheets is called a "skin-in-the-game" constraint. Given sufficient skin in the game, the exact composition of outside contracts is irrelevant. Indeed, once moral hazard problems are resolved between insiders and outsiders, the outside securities issued by insiders are indeterminate due to Modigliani-Miller holding on these securities. In particular, there are no taxes, costs of default, incomplete financial markets, or any other frictions that would violate MM, after agency problems are resolved. Therefore, the equity-like contract is without loss of generality under this interpretation.

Combining the assumptions above, insider net worth $n^z_{i,t}$ evolves as

$$dn^z_{i,t} = \underbrace{(n^z_{i,t} r_t - c^z_{i,t})dt}_{\text{consumption-savings}} + \underbrace{k^z_{i,t}(dR^z_{i,t} - r_t dt)}_{\text{capital ownership}} - \underbrace{\phi_z k^z_{i,t}(d\tilde{R}^z_{i,t} - r_t dt)}_{\text{outside funding}}$$
$$+ \underbrace{n^z_{i,t}\theta^z_{i,t} \cdot (\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}, \quad z \in \{A, B\}. \tag{7}$$

Given the ability to frictionlessly trade aggregate risk but not the idiosyncratic risk of capital, one can think of the differential $k^z_{i,t}\mathbb{E}_t[dR^z_{i,t} - r_t dt - \phi_z(d\tilde{R}^z_{i,t} - r_t dt)]$ as a compensation for idiosyncratic risk. Mathematically, households solve

$$\max_{n^z_i, c^z_i, k^z_i, \theta^z_i} \mathcal{U}^z_{i,t}, \quad z \in \{A, B\}, \tag{8}$$

subject to (7), $n^z_{i,t} \geq 0$, $k^z_{i,t} \geq 0$, where $\mathcal{U}^z_{i,t}$ is given by the logarithmic utility function (1).

Financier Problem

Financiers serve a diversification and safe-asset-creation role. Financiers hold a partially diversified portfolio of equity in each of the two sectors. They fund these activities by borrowing in riskless debt and using their own net worth.

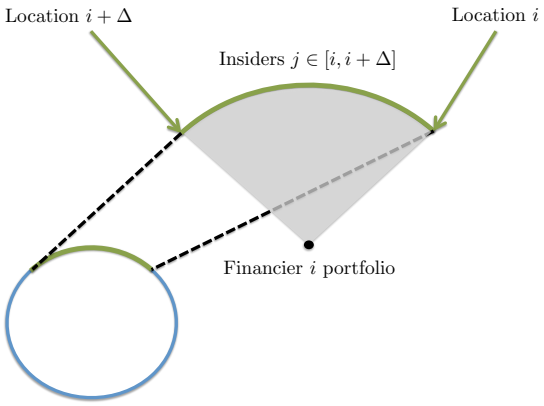

Figure 4: Circle of locations and financiers' partially diversified portfolios. Financiers have potentially different diversification parameters Δ_A and Δ_B for each sector.

I model diversification as follows. Financiers are tied to locations, just as insiders are. A financier located at $i \in [0, 1]$ invests in a portfolio of insiders' securities located "nearby" in the sense that they lie in a connected interval adjacent to location i. Define $\Delta_z \in [0, 1]$ to be the length of this interval for

For example, we may think of changing the degree of risk in the outside contract by setting

$$d\tilde{R}^z_{i,t} := (r_t + s^z_{i,t})dt + \zeta(dR^z_{i,t} - \mathbb{E}_t[dR^z_{i,t}]),$$

for $\zeta \leq 1$. The parameter ζ might capture the fact that insiders empirically borrow in debt which is less risky than the underlying asset. If $\zeta = 1$, the contract is equity. As $\zeta \to 0$, the contract approaches riskless debt. However, the parameter ζ is irrelevant in the following sense. One can verify that $\phi_z \zeta$ enters all formulas multiplicatively, and so enters all equilibrium expressions multiplicatively. What matters is that $\phi_z \zeta$ of risk is sold off to outsiders.

insiders in sector $z \in \{A, B\}$. Insiders financed by finanicer i are those with $j \in [i, i+\Delta_z]$ mod $[0,1]$. Here, the Δ_z are exogenously fixed numbers, not choices by financiers. I explore endogenous Δ_z choices in Appendix C.2. This partial but imperfect diversification arc on the circle may be visualized in figure 4.

For simplicity, I assume financiers fund all insiders within their investment arc symmetrically. Let $\lambda^z_{i,t}$ represent location-i financiers' funding, per unit of their net worth, of sector-z insiders. In other words, location-i financiers supply $\lambda^z_{i,t}\Delta_z^{-1} n^F_{i,t}$ of funds to each sector-z insider $j \in [i, i+\Delta_z]$ mod $[0,1]$, rather than allowing $\lambda^z_{i,t}$ to also vary by destination.[20]

Putting everything together, the financier's net worth evolves dynamically as follows:

$$dn^F_{i,t} = \underbrace{(n^F_{i,t} r_t - c^F_{i,t})dt}_{\text{consumption-savings}} + \underbrace{n^F_{i,t}\theta^F_{i,t} \cdot (\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}$$
$$+ \underbrace{\lambda^A_{i,t} n^F_{i,t} \Delta_A^{-1} \int_i^{i+\Delta_A} (d\tilde{R}^A_{j,t} - r_t dt) dj}_{\text{funding portfolio (sector } A)} + \underbrace{\lambda^B_{i,t} n^F_{i,t} \Delta_B^{-1} \int_i^{i+\Delta_B} (d\tilde{R}^B_{j,t} - r_t dt) dj}_{\text{funding portfolio (sector } B)}. \tag{9}$$

Financiers solve

$$\max_{n^F_i, c^F_i, \lambda^A_i, \lambda^B_i, \theta^F_i} \mathcal{U}^F_{i,t} \tag{10}$$

subject to (9), $n^F_{i,t} \geq 0$, $\lambda^A_{i,t} \geq 0$, $\lambda^B_{i,t} \geq 0$, where $\mathcal{U}^F_{i,t}$ is given by the logarithmic utility function (1).

Free Mobility

At this point, I make an important technical assumption that keeps the equilibrium construction tractable. Specifically, I assume a free-mobility condition between locations, which allows us to study a symmetric equilibrium.[21]

Assumption 2 (Free Mobility)**.** *Insiders and financiers are freely mobile among locations i.*

Under Assumption 2, idiosyncratic shocks will wash out in aggregate, but the expectation that they will hit matters for individual behavior. A similar free-mobility assumption has been used across the idiosyncratic "islands" of Gertler and Kiyotaki (2010). For details on an equilibrium of a similar model without Assumption 2, see Khorrami (2018). In that setting, a symmetric equilibrium is not possible. Instead, the entire distribution of net worth across locations becomes a state variable.

[20] Relaxing this assumption does not change the results significantly. With the maintained symmetry assumptions, $\{\lambda_{i\to j,t}\}$ could be chosen in two stages (if spreads $s_{j,t}$ are independent of location j). First, leverage $\lambda_{i,t} := \Delta^{-1}\int_i^{i+\Delta} \lambda_{i\to j,t} dj$ could be chosen to trade off return and risk, as in the main text. Second, risky share allocations $\lambda_{i\to j,t}/\lambda_{i,t}$ could be chosen to minimize the portfolio variance of a unit investment. One can verify that the resulting portfolio is exactly a symmetrically funded portfolio, with point masses on the extremal locations i and $i+\Delta$.

[21] Unlike location choices, agents' occupations (e.g., "sector A insider"; "financier") are fixed and cannot be chosen.

2.2 Equilibrium

Definition 1. *An* equilibrium *consists of price and allocation processes, adapted to the aggregate and idiosyncratic shocks* $\{(Z_t^A, Z_t^B, W_{i,t}^A, W_{i,t}^B) : i \in [0,1], t \geq 0\}$*, such that all agents solve their optimization problems and all markets clear. Prices consist of the interest rate* r_t*, aggregate risk price* π_t*, and spreads* $s_{i,t}^A, s_{i,t}^B$*. Allocations consist of capital and equity holdings* $(k_{i,t}^A, k_{i,t}^B, \lambda_{i,t}^A, \lambda_{i,t}^B)$*, consumption choices* $(c_{i,t}^A, c_{i,t}^B, c_{i,t}^F)$*, and aggregate risk hedging choices* $(\theta_{i,t}^A, \theta_{i,t}^B, \theta_{i,t}^F)$*. A* symmetric equilibrium *is an equilibrium in which all objects are independent of* i *for each* t*. The market-clearing conditions at every point in time are as follows.*

- Goods markets:

$$\int_0^1 [G_A k_{i,t}^A + G_B k_{i,t}^B] di dt = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F] di dt + \int_0^1 [dI_{i,t}^A + dI_{i,t}^B] di.$$

- Funding markets:

$$\int_{i-\Delta_z}^{i} \Delta_z^{-1} \lambda_{j,t}^z n_{j,t}^F dj = \phi_z k_{i,t}^z, \quad \forall i \in [0,1], \quad z \in \{A, B\}.$$

- Aggregate risk market:

$$\int_0^1 [\theta_{i,t}^A n_{i,t}^A + \theta_{i,t}^B n_{i,t}^B + \theta_{i,t}^F n_{i,t}^F] di = 0.$$

- Bond market:

$$\int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F] di = \int_0^1 [k_{i,t}^A + k_{i,t}^B] di.$$

First, we have a lemma which shows that, under additional restrictions, any equilibrium will be "location invariant" in a certain sense.

Lemma 2.3 (Location Invariance)**.** *Let Assumptions 1 and 2 hold. If an equilibrium is such that*

$$\frac{n_{i,t}^A}{n_{i,t}^A + n_{i,t}^B} \quad \text{and} \quad \frac{n_{i,t}^F}{k_{i,t}^A + k_{i,t}^B} \quad \text{and} \quad k_{i,t}^A + k_{i,t}^B$$

are independent of i*, that equilibrium must be location invariant in the sense that* $k_{i,t}^A/k_{i,t}^B$*,* $k_{i,t}^A/n_{i,t}^A$*,* $k_{i,t}^B/n_{i,t}^B$*,* $\lambda_{i,t}^A$*,* $\lambda_{i,t}^B$*,* $s_{i,t}^A$*, and* $s_{i,t}^B$ *are independent of* i*. Furthermore, a symmetric equilibrium is feasible.*

Among such location-invariant equilibria, I analyze the special one in which locations are exactly identical in their net worths, which is feasible (and weakly optimal) under free-mobility. Studying this equilibrium allows me to avoid keeping track of the full distribution of wealth among locations, which would otherwise be necessary to know the evolution of aggregates such as wealth.

For construction of the symmetric equilibrium, define aggregate capital $K_t := \int_0^1 [k_{i,t}^A + k_{i,t}^B] di$ and the capital distribution $\kappa_t := K_t^{-1} \int_0^1 k_{i,t}^A di$. Define the wealth shares

$$\alpha_t := \frac{N_{A,t}}{N_{A,t} + N_{B,t}} \quad \text{and} \quad \eta_t := \frac{N_{F,t}}{N_{F,t} + N_{A,t} + N_{B,t}},$$

where $N_{A,t} := \int_0^1 n_{i,t}^A di$, $N_{B,t} := \int_0^1 n_{i,t}^B di$, and $N_{F,t} := \int_0^1 n_{i,t}^F di$ are aggregate net worths. The only state variables in a symmetric equilibrium will be (α_t, η_t, K_t). Therefore, in what follows, I drop location i subscripts from all variables whenever the meaning is clear. All stationary variables will be solely functions of (α_t, η_t), whereas growing variables will form a stochastic trend around K_t. The state dynamics are

$$\begin{aligned} d\alpha_t &= \mu_t^\alpha dt + \sigma_t^\alpha \cdot dZ_t \\ d\eta_t &= \mu_t^\eta dt + \sigma_t^\eta \cdot dZ_t \\ dK_t &= K_t[\iota_t dt + \sigma_t \cdot dZ_t], \end{aligned}$$

where the aggregate investment rate ι_t is determined from $\iota_t K_t dt := dI_t^A + dI_t^B$ and the aggregate diffusion vector is given by $\sigma_t := \kappa_t \sigma_A + (1 - \kappa_t)\sigma_B$.[22] The equilibrium is computed explicitly.

Proposition 2.4 (Two-Sector Equilibrium)**.** *Let Assumptions 1 and 2 hold. Then, there exists a unique symmetric equilibrium with state variables (α, η). The equilibrium is non-stochastic in the sense that $\sigma^\alpha \equiv \sigma^\eta \equiv 0$. The state drifts are*

$$\mu^\alpha = \alpha(1-\alpha)[\hat{\pi}_A^2 - \hat{\pi}_B^2] \tag{11}$$

$$\mu^\eta = \eta(1-\eta)[\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2 - \alpha\hat{\pi}_A^2 - (1-\alpha)\hat{\pi}_B^2], \tag{12}$$

where

$$\hat{\pi}_A := \frac{\kappa(1-\phi_A)\hat{\sigma}_A}{\alpha(1-\eta)} \quad \textit{and} \quad \hat{\pi}_B := \frac{(1-\kappa)(1-\phi_B)\hat{\sigma}_B}{(1-\alpha)(1-\eta)} \tag{13}$$

$$\hat{\pi}_{F\to A} := \frac{\kappa\phi_A(1-\Delta_A)\hat{\sigma}_A}{\eta} \quad \textit{and} \quad \hat{\pi}_{F\to B} := \frac{(1-\kappa)\phi_B(1-\Delta_B)\hat{\sigma}_B}{\eta} \tag{14}$$

are shadow idiosyncratic risk prices. The aggregate risk price vector is $\pi = \kappa\sigma_A + (1-\kappa)\sigma_B$. The capital distribution is given by

$$\kappa = \min(1, \max(0, \tilde{\kappa})), \quad \textit{where} \tag{15}$$

$$\tilde{\kappa} := \frac{G_A - G_B + \|\sigma_B\|^2 + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2}{\|\sigma_A\|^2 + \|\sigma_B\|^2 + \left[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\right]\hat{\sigma}_A^2 + \left[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\right]\hat{\sigma}_B^2}.$$

[22] Note that, although sectoral investment dI_t^A and dI_t^B will not, in general, be absolutely continuous with respect to time (Lebesgue measure), the sum must be absolutely continuous as a consequence of goods market clearing.

Finally, the growth rate ι and interest rate r are given by

$$\iota = \kappa G_A + (1-\kappa) G_B - \rho \tag{16}$$

$$r = \rho + \iota - \|\pi\|^2 - (1-\eta)[\alpha \hat{\pi}_A^2 + (1-\alpha)\hat{\pi}_B^2] - \eta[\hat{\pi}_{F\to A}^2 + \hat{\pi}_{F\to B}^2]. \tag{17}$$

I should make a few preliminary comments on the equilibrium in Proposition 2.4. First, even though the two sectors produce the same consumption good, each sector can receive a non-trivial allocation of resources because of their risk properties. Indeed, the sectoral shocks (Z^A, W^A) and (Z^B, W^B) are independent, so it is efficient to diversify these shocks by producing some output in each sector. The same qualitative insights survive in a model with differentiated goods (imperfect substitutability), which provides an additional rationale for production diversification. See Appendix C.3 for the model with Cobb-Douglas preferences over the consumption goods.

Second, the expected excess return on each capital stock can be decomposed into the aggregate risk premium, plus idiosyncratic risk premia earned by insiders and financiers. These idiosyncratic risk premia are non-trivial due to imperfect diversification by both insiders (who must hold $1-\phi$ fraction of their capital risk) and financiers (who can only diversify Δ fraction of the locations). Indeed, for sectors $z \in \{A, B\}$,

$$\underbrace{G_z - r}_{\text{total risk premium}} = \underbrace{\sigma_z \cdot \pi}_{\text{agg risk premium}} + \underbrace{(1-\phi_z)\hat{\sigma}_z \hat{\pi}_z}_{\text{insiders' idio risk premium}} + \underbrace{\phi_z(1-\Delta_z)\hat{\sigma}_z \hat{\pi}_{F\to z}}_{\text{financiers' idio risk premium}}. \tag{18}$$

For example, $\sigma_z \cdot \pi$ is the aggregate risk premium in sector z, as the product of the quantity of risk (loading on dZ_t) and the aggregate risk price π. Similarly, the latter two terms represent idiosyncratic risk premia: $(1-\phi_z)\hat{\sigma}_z$ and $\phi_z(1-\Delta_z)\hat{\sigma}_z$ represent the quantity of idiosyncratic risk held by insiders and financiers, respectively, and $\hat{\pi}_z$ and $\hat{\pi}_{F\to z}$ are the prices of these risks. These idiosyncratic risk prices measure the marginal utility response to a negative idiosyncratic shock.

In addition, the non-stochastic nature of the economy is due to the combination of identical risk preferences and complete markets over aggregate risk. In particular, agents may frictionlessly pick their level of exposure to dZ_t, given the existence of a hedging securities market (i.e., $\theta_{i,t}^A$, $\theta_{i,t}^B$, and $\theta_{i,t}^F$ are unconstrained). I will relax this frictionless hedging in the quantitative application below.

Finally, because the state variables are deterministic, a reasonable conjecture is that the system eventually reaches a "steady state" as $t \to \infty$. This is the subject of the following proposition.

Proposition 2.5 (Steady State)**.** *In the equilibrium of Proposition 2.4, if initial wealth shares $\alpha_0, \eta_0 > 0$, and if $G_A - G_B$ is sufficiently small, then there exists a steady state given by $(\alpha_\infty, \eta_\infty)$, where*

$$\alpha_\infty := \frac{\kappa_\infty(1-\phi_A)\hat{\sigma}_A}{\kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B} \tag{19}$$

$$\eta_\infty := \frac{\sqrt{(\kappa_\infty \phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-\kappa_\infty)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2}}{\sqrt{(\kappa_\infty \phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-\kappa_\infty)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2} + \kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B}, \tag{20}$$

and where κ_∞ is given by the time-limit of equation (15).

The equilibrium from Propositions 2.4 and 2.5 can be conveyed graphically. The left panel of figure 5 plots the supply and demand in sector A's lending market, with the idiosyncratic risk price $\hat{\pi}_{F\to A}$ against the financier portfolio λ^A.

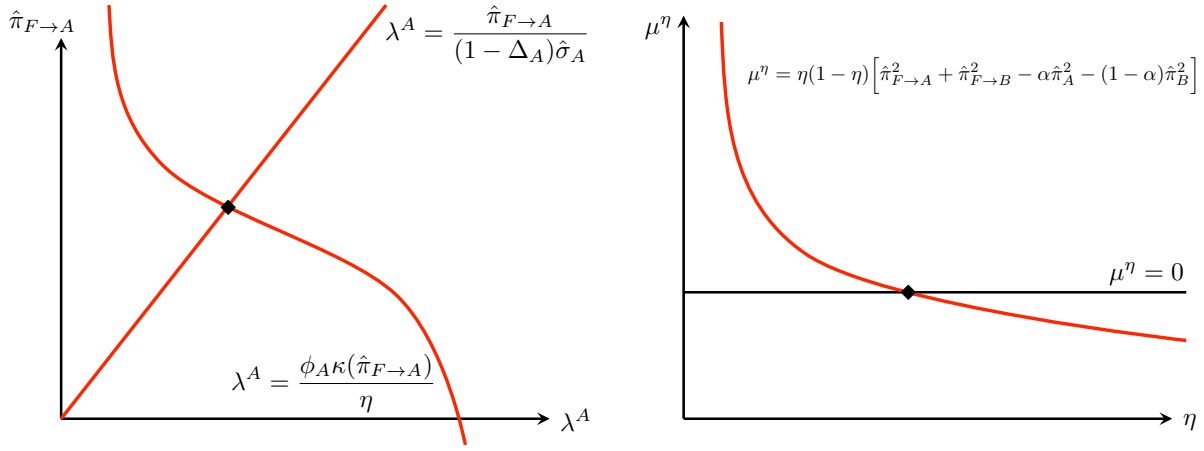

Figure 5: Steady-state equilibrium.

The increasing line is funding supply: financiers' optimal portfolio λ^A is simply a mean-variance portfolio trading off idiosyncratic risk compensation, $\hat{\pi}_{F\to A}$, against the idiosyncratic volatility of the portfolio, $(1-\Delta_A)\hat{\sigma}_A$. This investment can be chosen using only idiosyncratic risk considerations because of the frictionless market for trading aggregate risk.

The downward-sloping curve plots funding demand, which is constructed from insiders' optimal capital choice. Sector A capital demand, relative to aggregate capital, is

$$\kappa = \frac{G_A - r - \phi_A s_A - (1-\phi_A)\sigma_A \cdot \pi}{(1-\phi_A)^2 \hat{\sigma}_A^2}(1-\eta)\alpha.$$

Insiders retain $(1-\phi_A)$ of their capital risk as inside equity and optimally trade off its variance, $(1-\phi_A)^2\hat{\sigma}_A^2$, against its expected return. Inside equity earns the expected excess return $G_A - r$ on capital, net of the lending spread s_A paid to financiers on ϕ_A of outside equity. Because insiders are able to hedge aggregate risk inherent in capital ownership, they may additionally remove the aggregate risk premium on inside equity, $(1-\phi_A)\sigma_A \cdot \pi$. Because the spread s_A is fair, it compensates financiers for both aggregate and idiosyncratic risk:

$$s_A = \sigma_A \cdot \pi + (1-\Delta_A)\hat{\sigma}_A \hat{\pi}_{F\to A}.$$

All else equal, a higher risk price $\hat{\pi}_{F\to A}$ increases spreads s_A and lowers capital demand κ. Lower capital demand reduces funding demand through the equity-market-clearing relationship $\phi_A\kappa = \lambda^A\eta$, which is the downward-sloping curve plotted in figure 5.

The right panel shows the dynamics of η. The drift μ^η balances the relative profitabilities of financiers and insiders, which are governed by their idiosyncratic risk prices:

$$\mu^\eta = \eta(1-\eta)\Big[\underbrace{\hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}}_{\text{financier profitability}} - \underbrace{(\alpha\hat{\pi}_A + (1-\alpha)\hat{\pi}_B)}_{\text{insider profitability}}\Big].$$

As a function of η, μ^η is typically strictly decreasing, because $\hat{\pi}_{F\to A}$ and $\hat{\pi}_{F\to B}$ are decreasing in η, whereas $\hat{\pi}_A$ and $\hat{\pi}_B$ are increasing in η. This downward-sloping property is why the economy converges to the steady state, with $\mu^\eta = 0$, from any starting point.

2.3 Long-Run Effects of Better Diversification

In this section, I illustrate the reallocation and leverage effects discussed in the introduction. Suppose diversification improves in sector A, i.e., $\Delta_A \uparrow$. Figure 6 illustrates the adjustment of the economy to the new steady state.

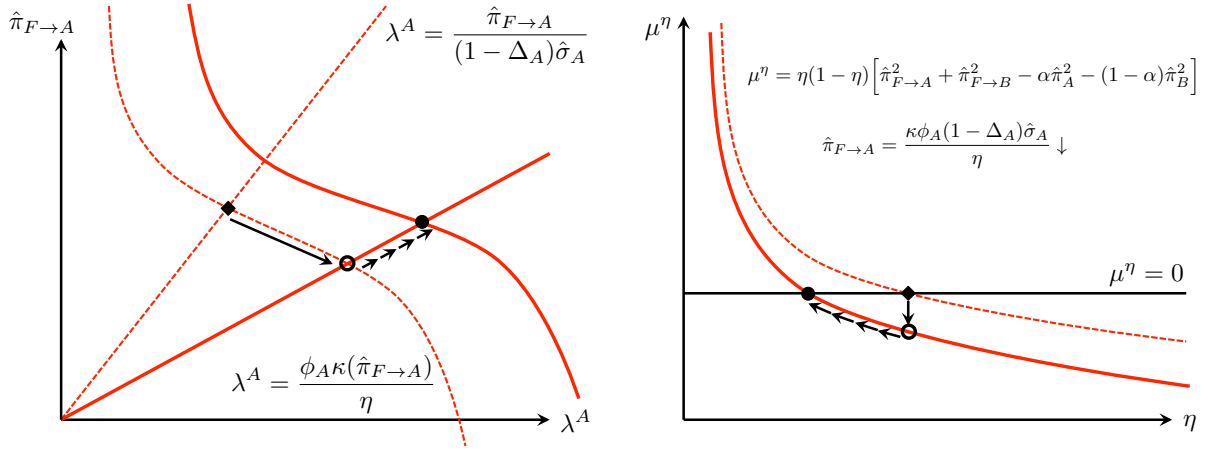

Figure 6: Equilibrium before and after an increase in diversification, $\Delta_A \uparrow$.

In the short run, better diversification increases sectoral funding supply because it improves financiers' risk-reward trade-off. Graphically, this improvement is captured by the outward rotation of the supply curve (left panel), which results in a shift from the diamond to the hollow circle.[23] This shift reduces equilibrium risk compensation $\hat{\pi}_{F\to A}$ and generates a discontinuous increase in the sector A capital share κ. Although aggregate capital will never jump in equilibrium, its sectoral allocation can, due to frictionless investment.

This short-run outcome is the *reallocation effect*. Diversification-induced reallocation can partly explain the stylized fact that sectoral capital shares are negatively correlated with sectoral risk premia, documented by Bansal, Ward and Yaron (2017). Reallocation can also occur with fundamental

[23] Note that there is a small outward shift in the supply on impact because equation (15) shows κ is increasing in Δ_A, independently of $\hat{\pi}_{F\to A}$. One can show this is second order relative to the shift in the supply curve, i.e., $\hat{\pi}_{F\to A}$ still falls on impact, holding η fixed.

shocks $dZ^A > 0$, but such fundamental reallocation tends to raise the sectoral risk premium through its aggregate component $\sigma_A \cdot \pi = \kappa \|\sigma_A\|^2$, exactly as in Cochrane, Longstaff and Santa-Clara (2007).

At the same time, lower risk compensation $\hat{\pi}_{F \to A}$ reduces financier profitability, so the drift μ^η shifts downwards (right panel). Over time, η drifts downwards. Financiers are happy to decumulate, because a lower quantity of idiosyncratic risk necessitates a lower precautionary savings buffer. However, lower relative wealth means financiers must accumulate leverage to continue their scale of financing operations. This dynamic effect is captured by the gradual outward shift in the demand curve (left panel), which results in a shift from the hollow circle to the solid circle. Because financiers are present in both sectors, a similar effect occurs in sector B's funding market.

I call this dynamic force the *leverage effect*. Indeed, financier leverage (assets/equity) is

$$\text{leverage} := \lambda^A + \lambda^B = \frac{\phi_A \kappa + \phi_B (1 - \kappa)}{\eta}, \tag{21}$$

so declines in η tend to increase leverage. The following result formalizes the preceding analysis.

Proposition 2.6 (Reallocation and Leverage)**.** *Consider the steady-state equilibrium of Proposition 2.5, with $G_A = G_B$, $\phi_A = \phi_B$, $\hat{\sigma}_A = \hat{\sigma}_B$, $\|\sigma_A\| = \|\sigma_B\|$, and $\Delta_A = \Delta_B$. The following hold:*

$$\begin{aligned}
&\textit{(i)} && \frac{d\kappa_\infty}{d\Delta_A} > 0 \quad \textit{and} \quad \frac{d\eta_\infty}{d\Delta_A} < 0; \\
&\textit{(ii)} && \frac{d\kappa_\infty}{d\Delta} = 0 \quad \textit{and} \quad \frac{d\eta_\infty}{d\Delta} < 0, \quad \textit{where} \quad \Delta := \Delta_A = \Delta_B; \\
&\textit{(iii)} && \frac{d\kappa_\infty}{d\Delta_A} > 0 \quad \textit{and} \quad \frac{d\eta_\infty}{d\Delta_A} = 0, \quad \textit{if} \quad \phi_A = \phi_B = 1.
\end{aligned}$$

Part (i) of Proposition 2.6 demonstrates a case in which both reallocation and leverage effects are in play. The assumption of initially symmetric sectors is made to derive unambiguous comparative statics in this case.[24] Parts (ii) and (iii) demonstrate necessary conditions for the reallocation and leverage effects. Part (ii) shows some asymmetric diversification increase is necessary for the reallocation channel. Sectoral allocations are immune to a broad increase in diversification. Part (iii) shows some segmented markets (through borrowing limits) are necessary for the leverage channel. Without segmented markets between insiders and financiers, the insiders of both sectors have zero long-run wealth, meaning there can be no financial sector leverage.[25]

The reason η_t falls slowly over time is that profits take time to accumulate. Said differently, the absence of entry/exit between sectors slows convergence. When the relative profitability in

[24] Another particularly simple case is the model with $\phi_B = \Delta_B = 1$. Under these parameters, $\alpha_\infty = 1$ and

$$\eta_\infty = \frac{\phi_A (1 - \Delta_A)}{1 - \phi_A + \phi_A (1 - \Delta_A)} \quad \text{and} \quad \kappa_\infty = \frac{G_A - G_B + \|\sigma_B\|^2}{\|\sigma_A\|^2 + \|\sigma_B\|^2 + (1 - \phi_A \Delta_A)^2 \hat{\sigma}_A^2}.$$

Observe that part (i) of Proposition 2.6 is satisfied in this case, as long as $\kappa_\infty \in (0, 1)$.

[25] An alternative that preserves segmented markets but allows insiders unlimited equity issuance is to assume financiers' discount rate is greater than insiders', i.e., $\rho_F > \rho_A = \rho_B := \rho$, so that insiders have wealth in the long run. Under this assumption, one can show that κ_∞ is increasing in Δ_A and η_∞ is decreasing in Δ_A, even if $\phi_A = \phi_B = 1$.

one sector falls, entry/exit should occur until profitabilities are equalized. With free entry/exit, immediate convergence obtains, as stated in the proposition below.

Proposition 2.7 (Free Entry)**.** *Consider the equilibrium of Proposition 2.4. At every point in time, suppose all agents can freely choose their "occupations" in the following sense: agents can costlessly decide to be financiers, sector A insiders, or sector B insiders. Then, $(\alpha_t, \eta_t) = (\alpha_\infty, \eta_\infty)$ for all t.*

2.4 Dynamic Response to a Diversification Shock

In this section, I translate the graphical analysis of figure 6 into a time-path, or an "impulse response function" (IRF). See Appendix A.8 for more details on these IRFs and proofs of the lemmas below. Due to the tractability offered by logarithmic utility and frictionless physical investment, computing IRFs is not problematic. An important feature is that this model contains no "impact response."

Lemma 2.8. *There is no state variable impact response to an unanticipated shock to Δ_A or Δ_B at time t, i.e., $(\alpha_t, \eta_t) = (\alpha_{t-}, \eta_{t-})$.*

The key intuition for Lemma 2.8 is that portfolio holdings are pre-determined before a shock, so wealth can only jump if asset prices jump. But frictionless investment implies capital prices are always equal to one; in particular, they cannot jump.

In figure 7, I illustrate the time-path of a one-time unanticipated shock from Δ_A. The shock occurs at time $t = 0$ and the system has reached steady state at that time. The left panel shows the time-path for Δ_A. The middle and right panels illustrate the responses of κ_t and $\lambda_F^A + \lambda_F^B$ – the reallocation and leverage effects.

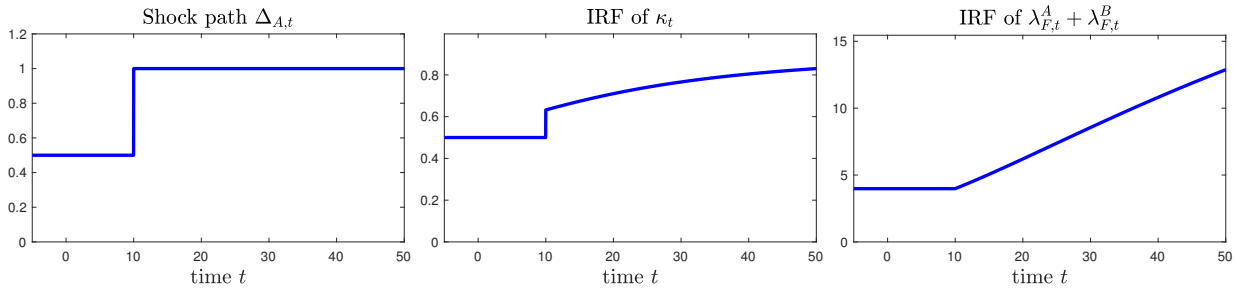

Figure 7: IRFs to a one-time shock from $\Delta_A = 0.5$ to $\Delta_A = 1$ at time $t = 0$. In this example, $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $G_A = G_B = 0.1$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $\rho = 0.02$, and $\Delta_B = 0.5$.

Alternatively, we also may want to consider a gradual increase in Δ_A. Performing this experiment raises the question of how to interpret repeated increases in Δ_A. One type of IRF treats the improvement in Δ_A as *unanticipated*, in the sense that economic agents perceive zero probability of diversification improvements, even though improvements repeatedly occur. This IRF can be computed by repeating the analysis of figure 7 with a series of smaller Δ_A shocks.

A second type of IRF treats the improvement as *fully anticipated*, in the sense that news about the future diversification path breaks at time τ, and after that time, agents know the entire future

time-path of diversification. A third type of IRF treats the improvement as *partially anticipated*, in the sense that agents know diversification levels follow a particular stochastic process. As shown by the lemma below, these three types of IRFs are equivalent.

Lemma 2.9. *Suppose either of the following holds:*

(i) $(\Delta_{A,t}, \Delta_{B,t})$ *follows a deterministic path. At time* τ*, agents are informed about a new future path* $\{(\Delta_{A,t}, \Delta_{B,t}) : t \geq \tau\}$.

(ii) $(\Delta_{A,t}, \Delta_{B,t})$ *follows an arbitrary Itô process. All agents are unconstrained in markets for Arrow claims on the shocks* $d\Delta_{z,t} - \mathbb{E}_t[d\Delta_{z,t}]$.

Then, the economy is in the equilibrium of Proposition 2.4 with $(\Delta_{A,t}, \Delta_{B,t})$ *representing* (Δ_A, Δ_B) *at every point in time* t.

As a consequence of Lemma 2.9, IRFs to a *series* of diversification shocks need not be interpreted as repeatedly fooling agents in the model with a sequence of zero-probability events. Instead, all we need to assume is either that agents are perfectly informed about the future diversification path, or that they can hedge future uncertainty to diversification paths. The crucial model feature for these lemmas is the optimally-myopic behavior of log utility agents, who care only about the current level of diversification, not its future probability distribution.

With these equivalence results in hand, the IRF from a gradual diversification improvement is displayed in figure 8, which is qualitatively a smoothed-out version of figure 7.

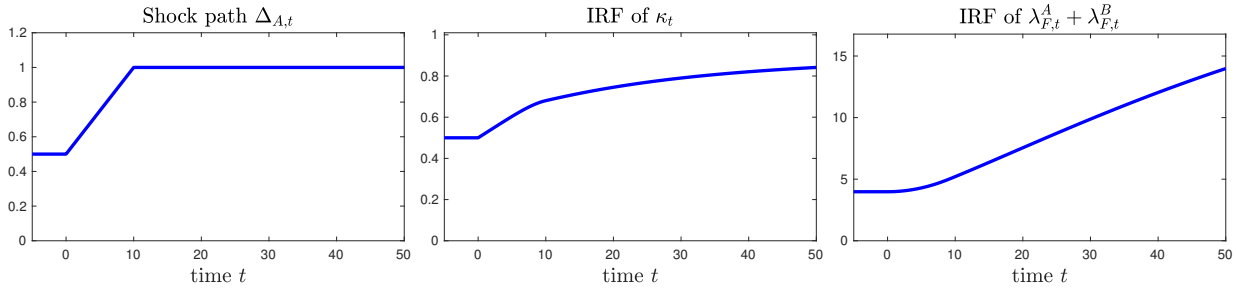

Figure 8: IRFs to a gradual increase from $\Delta_A = 0.5$ to $\Delta_A = 1$ from time $t = 0$ to $t = 10$. In this example, $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $G_A = G_B = 0.1$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $\rho = 0.02$, and $\Delta_B = 0.5$.

These time-paths connect the model to the 1990s-2000s housing boom. As shown in figure 1, this episode featured a large sectoral reallocation from corporate credit to household credit and a rise in financial intermediary leverage. Thus, if we interpret sector A as housing and sector B as productive capital, a gradual increase in Δ_A, corresponding to rising mortgage securitization or gradual banking deregulation, can match these qualitative patterns.

2.5 Lax Screening

The diversification-induced leverage build-up ("leverage effect") springs from a reduction in financier profitability, through lower equilibrium spreads. Empirically, there is some support for this during

the 2000s housing boom, as mortgage spreads fell and commercial bank profitability saw a modest decline.[26]

That said, the leverage build-up is a general response to better diversification and can be observed even in the absence of a profitability decline. For example, with better diversification, financiers initially reduce their lending spreads, so insiders may optimally borrow more by increasing their outside funding (i.e., ϕ_A and ϕ_B). Greater outside funding and better diversification have opposing effects on equilibrium spreads, so financier profitability may increase or decrease. At the same time, higher ϕ_A or ϕ_B directly raises financiers' assets/equity ratio by equation (21).

To analyze such a situation, Appendix A.5 generalizes the moral-hazard problem of insiders to generate the possibility of time-varying ϕ_A and ϕ_B. In this setup, the moral-hazard problem is smoothed in such a way that optimal short-term contracts cannot eliminate agency costs. Optimal issuance ϕ_A equates the marginal diversification benefits from offloading risk (arising because financiers are better-diversified than insiders) to marginal moral-hazard costs (arising because insiders will divert more resources when they keep less skin in the game). Improved financier diversification increases the marginal benefit of issuance, so ϕ_A rises with Δ_A. Although a lower skin-in-the-game requirement exacerbates insiders' agency problem, now-better-diversified financiers tolerate this cost. Credit standards are optimally relaxed, analogous to the story "securitization led to lax screening" in Keys, Mukherjee, Seru and Vig (2010). Thus, endogenizing credit standards demonstrates the leverage effect does not rely on falling financier profitability.[27]

3 Diversification-Induced Financial Crises

The model of Section 2 is meant to illustrate the reallocation and leverage mechanisms in a simple way. The key shortcoming of that model is the absence of any financial fragility: even though diversification reduces the financier wealth share, macroeconomic fluctuations are unaffected. Below, I modify the model to allow for the possibility of financial crises.

3.1 New Features

Leverage Constraints

First, I introduce a financier leverage constraint:

$$\lambda_{F,t}^A + \lambda_{F,t}^B \leq \bar{\lambda}, \qquad \bar{\lambda} > 1. \tag{22}$$

[26] See Justiniano et al. (2017) for evidence on mortgage spreads. See Appendix D.2 for commercial bank profitability.

[27] The dynamics of financier leverage are ambiguous in this story. If the net effect of higher Δ_A and higher ϕ_A is to reduce financier profitability, then η_t will fall over time, and leverage will continue to build. Otherwise, leverage will increase initially but decline over time.

Borrowing constraints like (22) can be a reduced-form for financier default costs that rise sharply with high leverage, or they may arise due to incentive problems.[28] In Appendix A.6, I micro-found the constraint with a simple agency problem. In that case, the endogenously-determined maximal leverage $\bar{\lambda}$ will in fact be a function of the economic state and model parameters. As I demonstrate, $\bar{\lambda}$ is not increasing in diversification Δ_A, Δ_B, which is particularly important for the analysis below.

The leverage constraint modifies portfolio choices by introducing an auxiliary variable (Lagrange multiplier) that I denote ζ_t. Thus, financiers' optimal portfolios are given by

$$\lambda_F^A = \frac{(s_A - \sigma_A \cdot \pi - \zeta)^+}{(1-\Delta_A)^2 \hat{\sigma}_A^2} \quad \text{and} \quad \lambda_F^B = \frac{(s_B - \sigma_B \cdot \pi - \zeta)^+}{(1-\Delta_B)^2 \hat{\sigma}_B^2}, \tag{23}$$

where $u^+ = \max(0, u)$. The standard complementary slackness condition determines when leverage constraints bind:

$$0 = \min\left\{\zeta, \bar{\lambda} - \lambda_F^A - \lambda_F^B\right\}. \tag{24}$$

The portfolio choices (23) are simple because the constraints (leverage and shorting) are homogeneous in wealth, and because all agents have log utility. See Appendix A.1 for a complete derivation using convex duality methods as in Cvitanić and Karatzas (1992). The presence of ζ helps us understand that a binding leverage constraint works similarly to a rise in intermediary funding costs. Indeed, as (23) suggests, the equilibrium with a leverage constraint is identical to an unconstrained economy in which financiers perceive a funding cost of $r + \zeta$ rather than r. Hence, I will sometimes refer to ζ as the "shadow funding cost."

Distressed Investors

Next, I introduce a fourth category of agent, which I call "distressed investors," who may also extend financing to insiders, but are less qualified to do so. In particular, for each unit of financing, distressed investors must pay a pecuniary cost χ out of their returns. Such costs may be a reduced-form for search costs, information-acquisition costs, fundraising costs, etc.[29] I implicitly assume these activities take time and other resources that would otherwise be used in production, so that this pecuniary cost is a deadweight loss to the economy. Although they are less skilled lenders, distressed investors do not face the leverage constraint (22). Finally, for quantitative purposes below, mainly to control average financier leverage, I assume financiers have a higher discount rate, $\rho_F > \rho$, than distressed investors and insiders. Otherwise, distressed investors are identical to financiers.

With distressed investors, equilibrium requires that we keep track of distressed investors' aggregate net worth $N_{D,t}$. In symmetric equilibrium, the wealth distribution is now characterized by three

[28] See, for example, Kehoe and Levine (1993), Hart and Moore (1994), Kocherlakota (1996), Kiyotaki and Moore (1997), Holmstrom and Tirole (1997), Gertler and Kiyotaki (2010), and Di Tella and Sannikov (2016).

[29] Distressed investors may also be interpreted as the marginal buyers of loans and securities on the secondary market, after the first-best financiers begin defaulting. For example, Chernenko, Hanson and Sunderam (2016) study mutual fund holdings of asset-backed securities, which grew in the financial crisis. Relatedly, He, Khang and Krishnamurthy (2010) study ABS holdings of hedge funds, commercial banks, and investment banks.

state variables:

$$\alpha := \frac{N_A}{N_A + N_B}, \quad \eta := \frac{N_F + N_D}{K}, \quad \text{and} \quad x := \frac{N_F}{N_F + N_D}.$$

Competition among insiders ensures distressed investors must also charge spreads s_A and s_B. Consequently, their return-on-investment is given by

$$\Delta_A^{-1} \int_i^{i+\Delta_A} (d\tilde{R}^A_{j,t} - \chi dt) dj + \Delta_B^{-1} \int_i^{i+\Delta_B} (d\tilde{R}^B_{j,t} - \chi dt) dj.$$

Their portfolio choices are given by

$$\lambda^A_D = \frac{(s_A - \sigma_A \cdot \pi - \chi)^+}{(1-\Delta_A)^2 \hat{\sigma}^2_A} \quad \text{and} \quad \lambda^B_D = \frac{(s_B - \sigma_B \cdot \pi - \chi)^+}{(1-\Delta_B)^2 \hat{\sigma}^2_B}. \tag{25}$$

Financial distress is said to occur when either spread rises beyond χ, such that $\lambda^A_D + \lambda^B_D > 0$.

Because the participation cost is modeled as a pecuniary cost, any equilibrium financial distress leads to inefficiency. The costs of financial distress appear in the modified resource constraint:

$$\underbrace{\iota + x\eta\rho_F + (1-x\eta)\rho}_{\text{Investment + Consumption}} = \underbrace{\kappa G_A + (1-\kappa) G_B}_{\text{Output}} - \underbrace{\chi\eta(1-x)(\lambda^A_D + \lambda^B_D)}_{\text{Costs of Financial Distress}}. \tag{26}$$

These costs are mechanically tied to periods of distress, although they scale with the degree of distress, i.e., the size of the costs depends on the level of participation by distressed investors.

Limited Hedging

Third, I introduce stochastic fluctuations into the economy by limiting aggregate risk hedging. In particular, I assume insiders cannot trade aggregate risk at all, i.e., $\theta^A_{i,t} \equiv \theta^B_{i,t} \equiv 0$. In reality, insiders of firms may be prevented from market trading due to incentive problems. Financiers and distressed investors may still trade aggregate risk with no constraints. This assumption generates stochastic fluctuations, because aggregate risk cannot be shared perfectly among agents.

Overlapping Generations

Lastly, I introduce a "perpetual youth" overlapping-generations (OLG) structure, to ensure a generically stationary wealth distribution, similar to Gârleanu and Panageas (2015). All agents perish independently at the Poisson rate δ. Since this assumption augments all agents' subjective discount rate by $+\delta$, parameters ρ and ρ_F should be thought of as inclusive of δ (see Lemma A.2). There are no markets to hedge these idiosyncratic death shocks. To keep the population size constant, newborns arrive at the same rate. Among newborns, the fraction entering sector z is ν_z, with $\nu_A + \nu_B + \nu_F + \nu_D = 1$. Dying agents' wealth is pooled and redistributed equally to newborns.

3.2 Equilibrium

Proposition 3.1 (Stochastic Equilibrium with Distress)**.** *Let Assumptions 1 and 2 hold, and augment financiers' problem with constraint (22). Let* $(\hat{\pi}_A, \hat{\pi}_B)$ *be insiders' idiosyncratic risk prices, defined in (13), and let* $(\hat{\pi}_{F\to A}, \hat{\pi}_{F\to B})$ *be financiers' idiosyncratic risk prices, defined by (14) with* $x\eta$ *in place of* η*. In a symmetric equilibrium,* (κ, ζ) *solve a nonlinear system given by (24) and*

$$0 = \min\left\{1-\kappa,\, H^+\right\} - \min\left\{\kappa,\, (-H)^+\right\} \tag{27}$$
$$H := G_A - G_B - \phi_A s_A + \phi_B s_B - (1-\phi_A)[\sigma_A\cdot\pi_A + \hat{\sigma}_A\hat{\pi}_A] + (1-\phi_B)[\sigma_B\cdot\pi_B + \hat{\sigma}_B\hat{\pi}_B],$$

where (s_A, s_B) *are equilibrium spreads,*

$$\begin{aligned} s_z - \sigma_z\cdot\pi = x\Big[(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z} + \zeta - \big(\zeta - \chi - \frac{x}{1-x}(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big] \\ + (1-x)\Big[\chi - \big(\chi - \zeta - (1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big], \quad z\in\{A,B\}, \end{aligned} \tag{28}$$

π is the traded aggregate risk price, and (π_A, π_B) *are insiders' shadow aggregate risk prices. If insiders may frictionlessly trade aggregate risk (unconstrained* θ_A, θ_B*), then* $\pi_A = \pi_B = \pi = \kappa\sigma_A + (1-\kappa)\sigma_B$*. If insiders may not trade (*$\theta_A = \theta_B \equiv 0$*), then* $\pi = \eta^{-1}[\kappa\phi_A\sigma_A + (1-\kappa)\phi_B\sigma_B]$*,* $\pi_A = \hat{\pi}_A\sigma_A/\hat{\sigma}_A$*, and* $\pi_B = \hat{\pi}_B\sigma_B/\hat{\sigma}_B$*. Define the following profitability functions:*

$$\Pi_A := \hat{\pi}_A^2 + \|\pi_A\|^2 \quad \textit{and} \quad \Pi_B := \hat{\pi}_B^2 + \|\pi_B\|^2 \tag{29}$$
$$\Pi_F := \lambda_F^A(s_A - \sigma_A\cdot\pi) + \lambda_F^B(s_B - \sigma_B\cdot\pi) + \|\pi\|^2 \tag{30}$$
$$\Pi_D := \lambda_D^A(s_A - \sigma_A\cdot\pi - \chi) + \lambda_D^B(s_B - \sigma_B\cdot\pi - \chi) + \|\pi\|^2. \tag{31}$$

State dynamics are given by

$$\mu^\alpha = \alpha(1-\alpha)[\Pi_A - \Pi_B] - (\alpha\pi_A + (1-\alpha)\pi_B)\cdot\sigma^\alpha + \delta((\nu_A+\nu_B)^{-1}\nu_A - \alpha) \tag{32}$$
$$\sigma^\alpha = \alpha(1-\alpha)[\pi_A - \pi_B], \tag{33}$$
$$\begin{aligned}\mu^\eta = \eta(1-\eta)[x(\rho-\rho_F) + x\Pi_F + (1-x)\Pi_D - \alpha\Pi_A - (1-\alpha)\Pi_B] \\ - (\eta\pi + (1-\eta)(\alpha\pi_A + (1-\alpha)\pi_B))\cdot\sigma^\eta + \delta(\nu_F + \nu_D - \eta)\end{aligned} \tag{34}$$
$$\sigma^\eta = \eta(1-\eta)[\pi - \alpha\pi_A - (1-\alpha)\pi_B], \tag{35}$$
$$\mu^x = x(1-x)[\rho - \rho_F + \Pi_F - \Pi_D] + \delta((\nu_F+\nu_D)^{-1}\nu_F - x) \tag{36}$$
$$\sigma^x = 0. \tag{37}$$

In Appendix A.3, Proposition A.3 states the analytical solution to this equilibrium by explicitly solving equations (24) and (27). An explicit solution is possible because the nonlinearity of this system is induced solely by the various portfolio constraints (i.e., leverage, shorting constraints). Such constraints bind on endogenous subsets of the state space, which I solve for analytically.

Figure 9 illustrates several properties of the equilibrium from Proposition 3.1. When η and x are

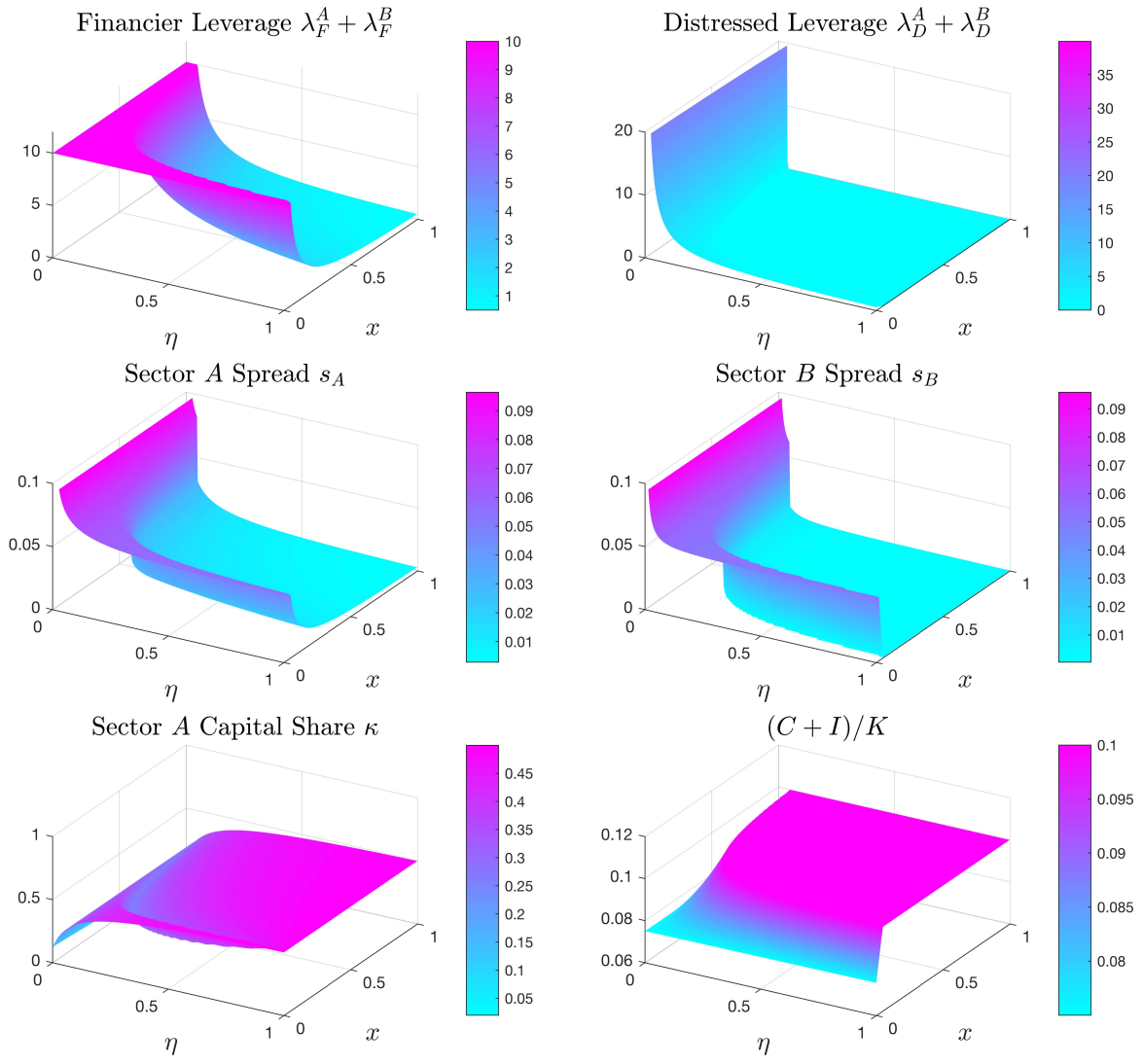

Figure 9: Equilibrium functions of (η, x) with $\alpha = 0.5$ fixed. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_A = 0.5$, $\Delta_B = 1$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = 10$.

low, financiers hit their leverage constraints. In this region, *financial distress* emerges as distressed investors enter the market and begin lending. Financial distress generates a sharp increase in the spreads of both sectors, even in sector B where $\Delta_B = 1$. The real effects of financial distress are summarized in the bottom-right panel, which shows a large decline in consumption plus investment relative to total capital, as expected from the resource constraint, equation (26).

3.3 Endogenous Bust and Financial Instability

The effects of a sectoral diversification improvement (increase in Δ_A) are summarized by the IRFs in figure 10. Importantly, Lemmas 2.8-2.9 continue to hold in this stochastic economy with leverage constraints. Thus, these IRFs can be interpreted as responses to anticipated improvements rather

than a series of zero-probability events.

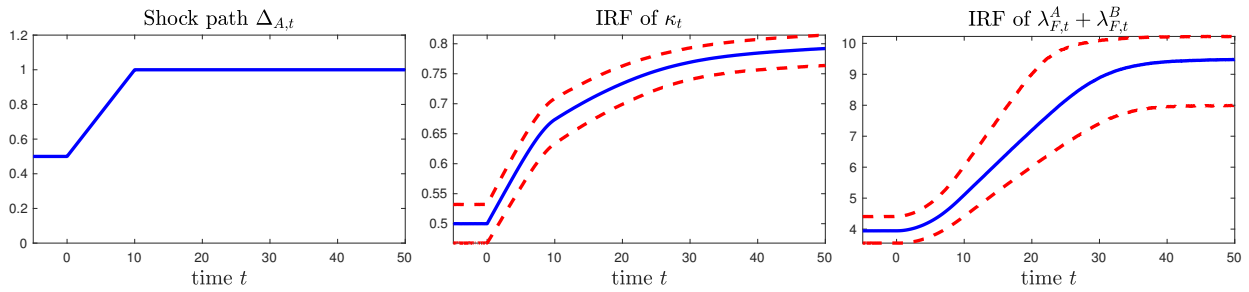

Figure 10: IRFs to a gradual increase from $\Delta_A = 0.5$ to $\Delta_A = 1$ from time $t = 0$ to $t = 10$. Solid lines are median responses, and dashed lines are 5th and 95th percentile responses. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_B = 0.5$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = 10$.

The responses are similar to those in figure 8 of Section 2.4, with two differences. First, the stochastic state dynamics induced by imperfect hedging imply a distribution of responses. I plot the 5th, 50th, and 95th percentile responses in figure 10. Second, the leverage constraint prevents an unmitigated rise in financier leverage.

A stark finding in this setting is the possibility of an *endogenous bust* in response to a diversification improvement. We say an endogenous bust occurs if there is a predicted future decrease in investment or consumption, relative to capital, in response to a shock at time τ, i.e.,

$$\exists t \geq 0 : \quad \mathbb{E}_{\tau-}[Y_{\tau+t} - Y_{\tau-}] < 0, \quad \text{where} \quad Y := \iota + x\eta\rho_F + (1 - x\eta)\rho. \tag{38}$$

Notice, from the resource constraint (26), investment and consumption are equal to "endogenous productivity" $\kappa G_A + (1-\kappa)G_B$ net of "distress costs" $\chi\eta(1-x)(\lambda^A_D + \lambda^B_D)$. For theoretical clarity, I will temporarily focus on an economy with symmetric sectors (in particular, $G_A = G_B$), so that the endogenous productivity term is constant. Furthermore, I will temporarily focus on a deterministic environment to easily identify the presence of an endogenous bust.

Proposition 3.2. *Consider the equilibrium of Proposition 3.1, with symmetric sectors and deterministic dynamics, i.e.,* $\nu_A = \nu_B$, $G_A = G_B \equiv G$, $\phi_A = \phi_B \equiv \phi$, $\Delta_A = \Delta_B \equiv \Delta$, $\hat{\sigma}_A = \hat{\sigma}_B \equiv \hat{\sigma}$, *and* $\sigma_A = \sigma_B = 0$. *Suppose there is a one-time increase* $\Delta_\tau - \Delta_{\tau-} > 0$ *at time* τ *and that* $\alpha_0 = 1/2$. *If* Δ_τ *is large enough, if* $(1-\phi/\bar{\lambda})^2(\bar{\lambda}\chi + \rho - \rho_F) + (1-\phi)^2\hat{\sigma}^2 < 0$, *and if*

$$(1-\phi/\bar{\lambda})^2(\bar{\lambda}\chi + \rho - \rho_F) + \frac{2\delta(1-\phi/\bar{\lambda})}{\nu_F + \nu_D}[(\nu_F\bar{\lambda}/\phi)^{1/2} - \nu_F - \nu_D] - (1-\phi)^2\hat{\sigma}^2 < 0 < \chi, \tag{39}$$

then an endogenous bust occurs, i.e., (38) holds.

Intuitively, Proposition 3.2 says the following. If diversification improves enough, then financiers will deterministically hit their leverage constraint in finite time. At that time, distressed investors will begin participating which triggers two effects: (i) a potential loss in output through "distress costs" that scale with participation; and (ii) a sudden increase in expected returns that rebuilds

financier wealth and pushes them away from the leverage constraint. If the rebuilding effect (ii) is sufficiently weak, then the economy gets stuck at the leverage constraint, inducing non-negligible participation by distressed investors, hence non-negligible distress costs. Condition (39) ensures a weak rebuilding effect.

A diversification improvement can generate *financial instability* as well as an endogenous bust. One measure of instability is the sensitivity of investment and consumption, relative to capital, to fundamental shocks, i.e.,

$$\text{Var}_t[dY_t], \quad \text{where} \quad Y := \iota + x\eta\rho_F + (1 - x\eta)\rho. \tag{40}$$

When $\text{Var}_t[dY_t] > 0$, then investment and consumption are more volatile than capital. This instability is tightly associated with a binding leverage constraint.

Proposition 3.3. *Consider the equilibrium of Proposition 3.1, with $G_A = G_B \equiv G$, $\phi_A = \phi_B \equiv \phi$, and suppose Δ_A, Δ_B are large enough. Then, there is instability, i.e., $\text{Var}_t[dY_t] > 0$, if and only if the leverage constraint (22) binds.*

Instability shows up in both sectors' risk premia. Figure 11 shows the IRFs of lending spreads to an improvement in sector A diversification. Once leverage constraints begin to bind, a large right tail appears suddenly in both sectors. Even though the diversification improvement was sector-specific, the sudden possibility of extreme spreads affects both sectors, because distressed investors typically enter both sectors when financiers are leverage-constrained. This tail event, with spillovers to all sectors, resembles a financial crisis.

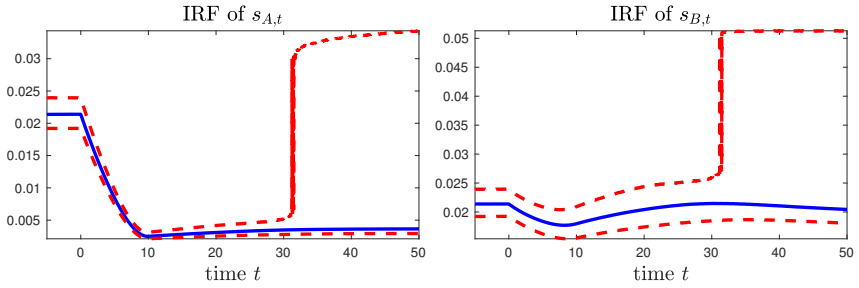

Figure 11: IRFs to a gradual increase from $\Delta_A = 0.5$ to $\Delta_A = 1$ from time $t = 0$ to $t = 10$. Solid lines are median responses, and dashed lines are 5th and 95th percentile responses. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_B = 0.5$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = 10$.

3.4 Necessity of Leverage Constraints

To see the crucial role the leverage constraint plays in the results above, now suppose $\bar{\lambda} = +\infty$. Figure 12 shows financial distress is almost completely absent. Distressed investors rarely enter the market, lending spreads respond much more smoothly to changes in the state variables, and sector B spreads are minuscule across the state space.

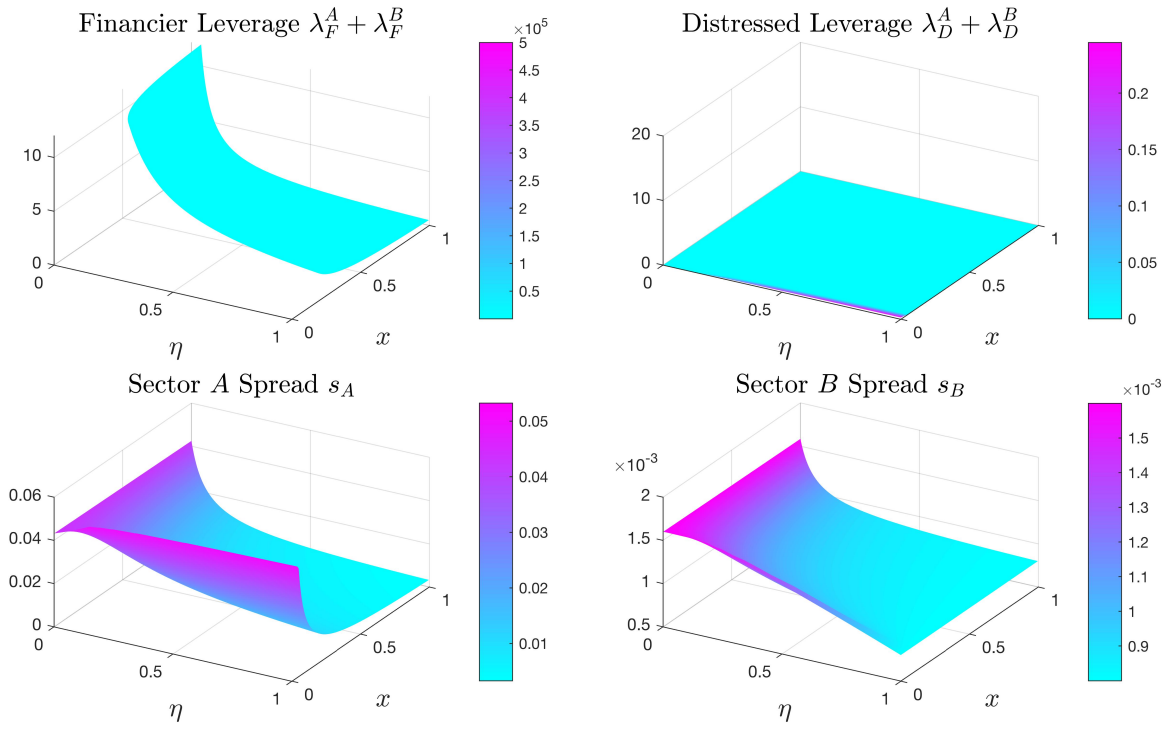

Figure 12: Equilibrium functions of (η, x) with $\alpha = 0.5$ fixed. Parameters: $\|\sigma_A\| = \|\sigma_B\| = 0.04$, $\hat{\sigma}_A = \hat{\sigma}_B = 0.20$, $\phi_A = \phi_B = 0.50$, $G_A = G_B = 0.1$, $\Delta_A = 0.5$, $\Delta_B = 1$, $\rho = 0.02$, $\rho_F = 0.06$, $\chi = 0.05$, and $\bar{\lambda} = +\infty$.

The following proposition formalizes this result by characterizing when financial distress occurs.

Proposition 3.4 (Distress without Leverage Constraints)**.** *Consider Proposition 3.1 with* $\bar{\lambda} = +\infty$*. Distressed investors lend to sector* $z \in \{A, B\}$ *if and only if financiers' wealth share* $x_t \eta_t < \omega_{z,t}^*$*, where*

$$\omega_{A,t}^* := \chi^{-1} \kappa_t \phi_A (1 - \Delta_A)^2 \hat{\sigma}_A^2 \tag{41}$$

$$\omega_{B,t}^* := \chi^{-1} (1 - \kappa_t) \phi_B (1 - \Delta_B)^2 \hat{\sigma}_B^2. \tag{42}$$

Proposition 3.4 illustrates the theoretical possibility of financial distress. If financiers' wealth is low relative to the amount of idiosyncratic risk they must bear, distressed investors have an incentive to enter the market. These incentives are summarized by the thresholds (ω_A^*, ω_B^*). That said, even for moderate diversification, these thresholds are tiny. Consider the case of symmetric sectors, such that $\kappa_t = 0.5$. Under $\chi = 0.05$ and $\hat{\sigma}_A = \hat{\sigma}_B = 0.2$, $\phi_A = \phi_B = 0.5$, and $\Delta_A = \Delta_B = 0.5$, we have $\omega_{A,t}^* = \omega_{B,t}^* = 0.05$. If financiers hold more than 5% of total wealth, distress is impossible.

Furthermore, as $\Delta_A, \Delta_B \to 1$, distressed investors never take positive positions, as (41)-(42) show. Under perfect diversification, financiers can perfectly hedge all the risks on their funding portfolio, so their leverage decisions are completely decoupled from the risks they must bear. Less efficient lenders never enter if they can finance more efficient lenders to do the same. This result explains why models without leverage constraints, such as Brunnermeier and Sannikov (2014), feature inefficiency that falls with financiers' fundamental risks.

Conversely, why does financial distress occur when $\bar{\lambda} < +\infty$? The following proposition characterizes distress in this case.

Proposition 3.5 (Distress with Leverage Constraints)**.** *In Proposition 3.1, the following hold:*

(i) If $\phi_A = \phi_B$*, then* $\lambda_{F,t}^A + \lambda_{F,t}^B = \bar{\lambda}$ *implies* $\lambda_{D,t}^A + \lambda_{D,t}^B > 0$*.*

(ii) Suppose $\chi \geq \bar{\lambda} \max\{(1-\Delta_A)^2 \hat{\sigma}_A^2, (1-\Delta_B)^2 \hat{\sigma}_B^2\}$*. Then,* $\lambda_{F,t}^A + \lambda_{F,t}^B < \bar{\lambda}$ *implies* $\lambda_{D,t}^A + \lambda_{D,t}^B = 0$*.*

Part (i) of Proposition 3.5 is a case in which distressed investors participate whenever (22) binds. Intuitively, because financiers are unable to raise new equity, they will be *forced* to de-lever upon hitting constraint (22), independent of their risk exposures and degree of hedging activities. De-leveraging automatically results in inefficient participation by distressed investors. In this sense, leverage constraints can introduce periods of financial distress.

Part (ii) says that, under certain parameterizations, distress can only occur if (22) binds. In this sense, distress is unlikely without leverage constraints. For example, under $\Delta_A = \Delta_B = 1$, this economy experiences distress if and only if leverage constraints bind.

4 Comparison to Other Shocks

Diversification improvements offer an answer to why booms are often sectoral (reallocation) and why sectoral booms may produce broad busts (financial leverage). In this section, I study several other "financial shocks" in my model: an LTV shock, a capital-requirement shock, a risk-tolerance shock, an uncertainty shock, and a foreign-savings shock. The motivation to study these shocks is that the extant literature has linked them at some point to boom-bust cycles, most recently related to the 2000s US housing boom. I show that, other than the diversification shock, none can produce both a sectoral reallocation and intermediary leveraging in my model. The results of this analysis are summarized in Table 1.

4.1 LTV Shock

Another important financial shock is an increase in ϕ, which reduces the idiosyncratic risk insiders must bear when investing in capital. Like a loan-to-value ratio, ϕ is the fraction of assets that insiders can borrow against, so I refer to this shock as an *LTV shock*. This type of shock is widely studied in the quantitative modeling literature, with somewhat disparate results.[30] Here, I study the implications of this shock in my model. We have the following result.

Proposition 4.1 (LTV Shock)**.** *Consider the equilibrium of Proposition 2.4, with* $G_A = G_B$*,* $\phi_A = \phi_B$*,* $\hat{\sigma}_A = \hat{\sigma}_B$*,* $\|\sigma_A\| = \|\sigma_B\|$*, and* $\Delta_A = \Delta_B$*. Then,* $\frac{d\eta_\infty}{d\phi_A} > 0$*, and if* Δ_A *is sufficiently large,* $\frac{d\kappa_\infty}{d\phi_A} > 0$*.*

[30] See, for example, Kiyotaki et al. (2011), Justiniano et al. (2015b), Favilukis et al. (2017), and Kaplan et al. (2017).

Financial Shocks	**Stylized Facts** Sectoral Reallocation	Financial Leverage
Diversification	$+$	$+$
LTV	$+$	$\sim$
Capital requirements	$\sim$	$+$
Financier risk-tolerance	$\sim$	$+$
Insider risk-tolerance	$+$	$-$
Idiosyncratic sectoral risk	$+$	$\sim$
Foreign savings	$\sim$	$+$

Table 1: Stylized facts and financial shocks. "$+$" indicates a positive response in the stylized fact to the referenced shock. "$-$" indicates a negative response in the stylized fact to the shock. "$\sim$" indicates a neutral or ambiguous response in the stylized fact to the shock.

The key point to note about ϕ is that it is a risk transfer between insiders and financiers. Because financiers are better diversified than insiders, this risk transfer is value-enhancing and generates sectoral reallocation. Mathematically, equation (18) shows that higher ϕ_A lowers sector A's idiosyncratic risk premia, which are equal to

$$\text{idio rp}_A = \kappa\Big[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\Big]\hat{\sigma}_A^2.$$

This quantity is decreasing in ϕ_A for a well-diversified sector.

That said, the risk transfer to financiers shifts idiosyncratic risk compensation from insiders to financiers. In response to the LTV shock, lending spreads increase, which is why LTV shocks are sometimes interpreted as "credit demand shocks." Thus, an increase in ϕ_A unambiguously raises financier profitability and their long-run wealth share. Although short-run financier leverage $\frac{\kappa\phi_A+(1-\kappa)\phi_B}{\eta}$ can increase with ϕ_A, the effect on long-run financier leverage is ambiguous through the rise in η.

4.2 Capital-Requirement Shock

Another possible finance-centric explanation for boom-bust cycles is improved financier access to outside equity. Perhaps financiers are equity-issuance constrained, perhaps because of capital requirements or more fundamental agency frictions. A relaxation in capital requirements improves financiers' risk-sharing with the rest of the economy. To model this scenario, I allow financiers to partially issue equity against their assets, requiring them to keep $1-\phi_F$ fraction of skin in the game, like the insiders of sectors A and B.[31] Shocks to the parameter ϕ_F can be called *capital requirement shocks*. We have the following result.

[31] This outside equity is assumed to be pooled, thus perfectly diversified, and sold to the market. The equilibrium of this modified economy is detailed in the appendix.

Proposition 4.2 (Capital-Requirement Shock)**.** *Consider the equilibrium of Proposition 2.4, with capital requirement* $1-\phi_F$*. If* $\Delta := \Delta_A \equiv \Delta_B$*, then* ϕ_F*-shocks and* Δ*-shocks are equivalent in the following sense: the equilibrium only depends on* $\Delta^* := 1-(1-\Delta)(1-\phi_F)$ *and not* ϕ_F *or* Δ *independently.*

Capital-requirement shocks (ϕ_F) are similar to diversification shocks (Δ) in that both provide ways for financiers to diversify idiosyncratic risks. For this reason, both parameters appear together in the expression for financiers' idiosyncratic risk prices, i.e.,

$$\hat{\pi}_{F\to A} = \frac{\kappa\phi_A(1-\Delta_A)(1-\phi_F)\hat{\sigma}_A}{\eta} \quad \text{and} \quad \hat{\pi}_{F\to B} = \frac{(1-\kappa)\phi_B(1-\Delta_B)(1-\phi_F)\hat{\sigma}_B}{\eta}.$$

Indeed, Proposition 4.2 shows that looser capital requirements act like broad, sectorally-agnostic increases in diversification. It follows that looser capital requirements will generate financial leverage. But a key distinction is that ϕ_F applies symmetrically to both sectors, whereas Δ_A, Δ_B can be asymmetric. Although a sector-specific diversification shock generates a reallocation, looser capital requirements will tend to raise asset prices and allocations across the board, as would a broad diversification improvement.

This is empirically relevant. Referring back to the motivational figure 1, we see household credit rose as a *share* of total private non-financial credit. From the multi-asset perspective, diversification shocks are more likely to generate these features than a general capital-requirement shock.

This is also relevant to theory. Justiniano et al. (2015a) adopt a reduced-form credit-supply shock, a relaxation of "lending constraints," as a plausible explanation for why house prices rose. But in that paper, the only positive net supply asset is housing, so lending constraints do indeed raise house prices. With multiple assets, house prices may rise with relaxed lending constraints, but they will rise in concert with other asset prices.

4.3 Risk-Tolerance Shock

A popular culprit of boom-bust cycles has been excessive optimism or excessive risk tolerance, e.g., Kindleberger (1978). Because of the nature of asset pricing, beliefs and risk tolerance always enter risk premia jointly. I thus consider shocks to risk tolerance in this section.

Now, agents are endowed with recursive utility as in equation (63) in the appendix. See Appendix A.2 for details on solving agents' portfolio problems under these preferences. For simplicity, I assume all agents have unitary elasticity of intertemporal substitution, but they differ in their risk-aversion parameters, $\gamma_A, \gamma_B, \gamma_F$. Risk-tolerance shocks are shocks to these parameters individually.

Proposition 4.3 (Risk-Tolerance Shock)**.** *Consider equilibrium with risk aversions* γ_A*,* γ_B*, and* γ_F*.*

(i) Suppose $G_A = G_B$*,* $\phi_A = \phi_B$*,* $\hat{\sigma}_A = \hat{\sigma}_B$*,* $\|\sigma_A\| = \|\sigma_B\|$*, and* $\Delta_A = \Delta_B$*. Suppose at time* t*,*

$\gamma_A = \gamma_B = \gamma_F$ and the economy is in steady state.[32] *The following hold:*

$$\begin{aligned}
&\textit{(insider risk-tolerance)} && \frac{d\kappa_t}{d\gamma_A^{-1}} > 0 \quad \textit{and} \quad \frac{d\mu^\eta(\eta_t)}{d\gamma_A^{-1}} > 0 \\
&\textit{(financier risk-tolerance)} && \frac{d\kappa_t}{d\gamma_F^{-1}} = 0 \quad \textit{and} \quad \frac{d\mu^\eta(\eta_t)}{d\gamma_F^{-1}} < 0 \\
&\textit{(both risk-tolerance)} && \frac{d\kappa_t}{d\gamma_A^{-1}} + \frac{d\kappa_t}{d\gamma_F^{-1}} > 0 \quad \textit{and} \quad \frac{d\mu^\eta(\eta_t)}{d\gamma_A^{-1}} + \frac{d\mu^\eta(\eta_t)}{d\gamma_F^{-1}} < 0.
\end{aligned}$$

(ii) Suppose the assumptions of part (i) hold, except $\hat{\sigma}_A > \hat{\sigma}_B = 0$. Then, $\frac{d\mu^\eta(\eta_t)}{d\gamma_A} + \frac{d\mu^\eta(\eta_t)}{d\gamma_F} = 0$.

Intuitively, an increase in γ_A^{-1} lowers discount rates in sector A, which generates a sectoral allocation. However, with lower discount rates, insiders are willing to pay higher spreads to financiers, increasing their long-run wealth share. In this sense, a γ_A-shock is a credit-demand shock, just like the LTV shock to ϕ_A. An increase in γ_F^{-1} is a credit-supply shock, because it lowers required lending spreads. But because γ_F applies symmetrically to both sectors, lending spreads decrease across the board. A sectoral reallocation is less likely, as with the capital requirement shock to ϕ_F. Only if γ_A^{-1} and γ_F^{-1} both increase, with γ_B^{-1} left unchanged, can the model generate both reallocation and leverage.[33] That said, the leveraging effect is muted by the fact that both lender and borrower idiosyncratic risk prices are reduced by the risk-tolerance shock. As part (ii) of Proposition 4.3 shows, this offsetting can be complete if the other sector has no idiosyncratic risk. Finally, one must ask what sprouted the sector-specific optimism, whereas diversification shocks are more readily measurable.

4.4 Uncertainty Shock

Uncertainty shocks have been proposed as a possible driver of cycles: when uncertainty is low, banks may take greater leverage, and the economy suffers when uncertainty reverts. A sectoral uncertainty shock would be a reduction in $\hat{\sigma}_A$. We have the following result, which shows that lower sectoral uncertainty generates a reallocation but may not generate financier leveraging.

Proposition 4.4 (Uncertainty Shock)**.** *Consider the equilibrium of Proposition 2.4, with $G_A = G_B$, $\phi_A = \phi_B$, $\|\sigma_A\| = \|\sigma_B\|$, and $\Delta_A = \Delta_B$. Suppose the economy is in steady state at time t. If $\hat{\sigma}_A = \hat{\sigma}_B$, then*

$$\frac{d\kappa_t}{d\hat{\sigma}_A} < 0 \quad \textit{and} \quad \frac{d\mu^\eta(\eta_t)}{d\hat{\sigma}_A} = 0.$$

If instead $\hat{\sigma}_A > \hat{\sigma}_B = 0$, then

$$\frac{d\kappa_\infty}{d\hat{\sigma}_A} < 0 \quad \textit{and} \quad \frac{d\eta_\infty}{d\hat{\sigma}_A} = 0.$$

[32] Note that with identical risk preferences, even if they are non-log preferences, and only fundamental shocks, the economy features a non-stochastic equilibrium that converges onto a balanced growth-path, for the same reasons as in Proposition 2.5.

[33] If sector A is interpreted as housing, such a shock corresponds most closely to the survey evidence in Case and Shiller (2003) and the evidence in Foote et al. (2012). Kaplan et al. (2017) and Glaeser and Nathanson (2017) have model economies where agents become optimistic only about housing. Even though Landvoigt (2016) incorporates securitization, a key element of his story is the underpricing of mortgage risk by lenders.

To understand this result, consider a hypothetical economy with no diversification ($\Delta_A = \Delta_B = 0$) but two values of idiosyncratic volatility that apply to insiders and financiers separately, i.e., $\hat{\sigma}_{A,A}$ and $\hat{\sigma}_{A,F}$. The economy is otherwise exactly identical. One can show the equilibrium of this economy is isomorphic to the equilibrium of Proposition 2.4, if $\hat{\sigma}_{A,A} = \hat{\sigma}_A$ and $\hat{\sigma}_{F,A} = (1-\Delta_A)\hat{\sigma}_A$. Therefore, a diversification shock operates by lowering $\hat{\sigma}_{F,A}$ and keeping $\hat{\sigma}_{A,A}$ fixed.

An uncertainty shock has the effect of lowering both $\hat{\sigma}_{F,A}$ and $\hat{\sigma}_{A,A}$ proportionally. The result of this type of shock is to scale down all agents' idiosyncratic risk premia equally. The long-run effect of low idiosyncratic uncertainty is ambiguous in the sense that η_t could be higher or lower, precisely because both insiders and financiers are affected.[34]

4.5 Foreign-Savings Shock

A final alternative to consider is an increase in demand for safe assets, which tends to reduce interest rates and may fuel the boom, e.g., Bernanke (2005). Because much of this safe-asset demand manifested empirically as foreign agents buying US Treasury securities and other close substitutes, I call this a *foreign-savings shock*. This is also consistent with the documented increase in foreign demand for highly-rated securitized products, which behave like safe assets.

To model foreign savings, I introduce a wedge into the bond-market-clearing condition, which now becomes

$$N_{A,t} + N_{B,t} + N_{F,t} + N_t^* = K_t.$$

I assume N_t^* follows some exogenous process. A foreign-savings shock can be modeled as an exogenous change to N_t^*. Note that foreign savings also affects the goods market, because net interest payments to foreigners must come out of consumption. This modified economy has three state variables, the relative wealth between financiers, insiders, and foreigners:

$$\alpha_t := \frac{N_{A,t}}{N_{A,t} + N_{B,t}}, \quad \eta_t := \frac{N_{F,t}}{N_{F,t} + N_{A,t} + N_{B,t}}, \quad \text{and} \quad \eta_t^* := \frac{N_t^*}{K_t}.$$

The equilibrium of this modified economy is detailed in the appendix. We have the following result.

Proposition 4.5 (Foreign-Savings Shock)**.** *Suppose* $\|\sigma_A\| = \|\sigma_B\|$*,* $\hat{\sigma}_A = \hat{\sigma}_B$*,* $\phi_A = \phi_B$*, and* $\Delta_A = \Delta_B$*. Suppose there is a one-time increase,* $\eta_t^* - \eta_{t-}^* > 0$*, in foreign savings. Suppose* $(\alpha_{t-}, \eta_{t-}, \kappa_{t-}) = (\alpha_\infty, \eta_\infty, 1/2)$ *prior to the shock. Then,* $\kappa_t = \kappa_{t-}$ *and* $\eta_t = \eta_{t-}$*.*

The key to Proposition 4.5 is that foreign inflows raise all domestic agents' leverage proportionally. Foreign savings of η_t^* per unit of domestic wealth result in leverage of $(1-\eta_t^*)^{-1}$ for the domestic representative agent. In particular, financier leveraging does occur after a foreign-savings shock.

[34] This speaks to an important difference between how I am modeling the financial sector and how it has been modeled in the literature. Because both financiers and insiders are taking idiosyncratic risks, they both demand idiosyncratic risk compensation that rises with higher uncertainty. In Appendix C.1, I show my way of setting up the model leads to substantively different conclusions about uncertainty shocks than Di Tella (2017). Indeed, I show uncertainty shocks do not lead to excessive intermediary risk concentration, because both insiders and financiers have negative hedging demands against high uncertainty states. From a deeper theoretical perspective, diversification shocks, which are uncertainty shocks aimed directly at the financial sector, are more likely to be a source of risk concentration.

But the leverage is distributed equally across all domestic agents. As a result, all idiosyncratic risk prices are given by the formulas in (13)-(14), with an additional scaling by $(1-\eta_t^*)^{-1}$. Formulas (11)-(12) then show the dynamics of (α, η) are merely scaled by $(1-\eta_t^*)^{-2}$, explaining why η_t is unaffected by foreign savings near the steady state. Applying this logic to formula (15) also explains why κ_t is unaffected by foreign savings near the steady state. Intuitively, foreign savings are not directed toward any particular sector, so reallocation does not occur.

5 Quantification: US Housing Cycle

Section 2 showed that better sectoral diversification generates a reallocation and financier leverage. The objective of this section is to quantify these effects in the context of the 1990s-2000s US housing cycle. The first step is to determine a reasonable size for the diversification shock (Section 5.1). The second step is to calibrate the model to fit this particular episode (Section 5.2). The validation of the model is judged by its ability to generate plausible dynamics for series not targeted by the calibration – financier leverage and lending spreads in both sectors. I show that the model without diversification improvements cannot even qualitatively generate the same dynamics.

5.1 Measuring Diversification

In this section, I construct a quantitative measure of mortgage-market diversification. At a high level, the steps involved are as follows. First, I construct synthetic mortgage portfolios for mortgage lenders, using originations data in the HMDA dataset. For loans that are sold or securitized, I assume they are 100% diversified. Loans that are held on the lender's balance sheet are imperfectly diversified, and computing the exact degree of diversification follows the instructions below. The result is therefore a holistic measure of diversification, accounting for loan sales to Fannie/Freddie, securitizations, and geographic diversification.

Second, I compute the one-year-ahead volatility of each lender's mortgage portfolio, using location-specific house-price changes as the proxy for each loan's return.[35] The lender's portfolio return is simply a weighted average of these loan-level returns, and I compute the volatility of this return. Importantly, this method automatically accounts for the empirical correlation between loans held on a lender's balance sheet. Denote the average lender-level volatility $\hat{\sigma}_{\Delta,t}$. Then, I proxy loan-level risk by measuring the average of all locations' one-year-ahead house-price volatility. Denote this average location-level volatility $\hat{\sigma}_t$. Finally, I back out time-varying diversification Δ_t using the model-implied relationship $(1-\Delta_t)\hat{\sigma}_t = \hat{\sigma}_{\Delta,t}$. Details on this procedure are in Appendix D.3.

[35] In doing this, I am assuming the risk on lender's mortgage portfolios can be proxied by the risk inherent in the house prices to which the mortgages are attached (or at least assuming the mortgage risk is proportional to the house-price risk). This proportionality assumption is incorrect per se, mainly because mortgages are debt contracts, which can be thought of as nonlinear functions of the local house price (e.g., default in bad states). But my assumption is reasonable as long as the covariances between the house prices in different locations are similar to the covariances between mortgage payments in different locations, because these covariances are the key inputs in how I measure diversification.

In figure 13, I plot the diversification index, Δ_t. In 1990, under 60% of housing risk was diversified by lenders.[36] By 2005, over 90% of such risk was diversified. During the same time period, the idiosyncratic volatility of housing ($\hat{\sigma}_t$) was not significantly reduced, indicating lenders faced lower housing risks primarily due to diversification.

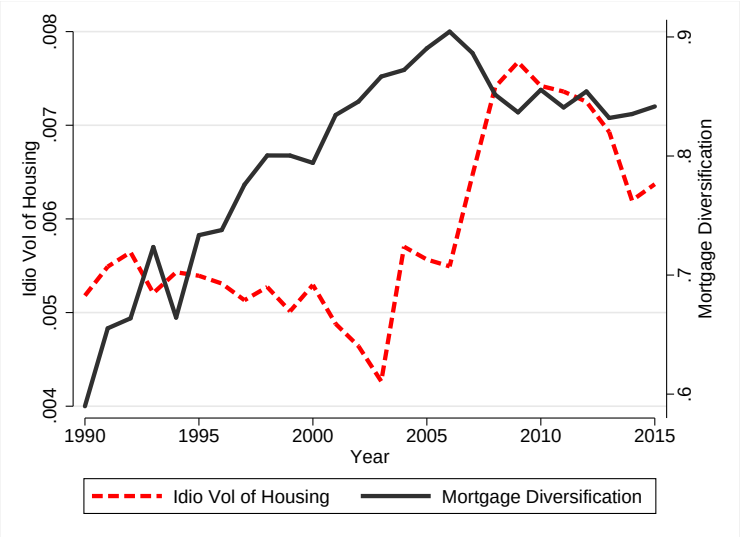

Figure 13: Diversification Index. "Idio Vol of Housing" plots estimates of $\hat{\sigma}_t$. "Mortgage Diversification" plots estimates of Δ_t. In this figure, the definition of "location" is a county. Source: HMDA and CoreLogic.

Why did diversification increase so dramatically? I find both securitization and geographic diversification were significant factors. Figure 14 shows the number of counties represented by loans in an average lender's portfolio increased from 10 to 30 during the boom. During the same time, the fraction of mortgage loans sold (either to Fannie/Freddie or to private-label securitizations) increased from 45% to 60%. The geographic diversification seems to have been under-appreciated during this episode.

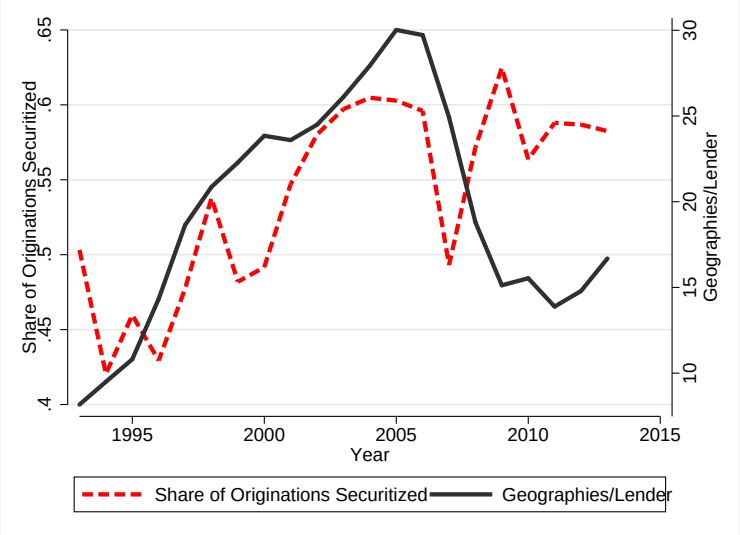

Figure 14: Diversification Components. "Share of Originations Securitized" shows the fraction of mortgage originations that are securitized within the same year of origination. "Geographies/Lender" computes the average number of counties per lender for loans held on lender's balance sheets. Source: HMDA.

[36] I do not go back to the 1980s because of HMDA data limitations. As discussed in Mian et al. (2017b) and Fieldhouse et al. (2018), banking deregulations and mortgage securitizations (by GSEs) began aggressively in the 1980s.

5.2 Calibrated Model

In this section, I interpret sector A as the housing sector, and sector B as all other productive capital. The parameters and targets for this model are listed in table 2.

Parameter		Value	Targets	
Panel A: Fundamentals				
G_A	productivity	0.04	housing average return	0.04
G_B	productivity	0.06	capital-housing wealth ratio*	3
$\|\sigma_A\|$	aggregate vol	0.03	aggregate house price vol	0.03
$\|\sigma_B\|$	aggregate vol	0.053	output growth vol*	0.04
$\hat{\sigma}_A$	idiosyncratic vol	0.11	idiosyncratic house price vol	0.11
$\hat{\sigma}_B$	idiosyncratic vol	0.25	idiosyncratic stock price vol	0.25
Panel B: Preferences / OLG				
ρ	discount rate	0.02	riskless rate*	0.02
ρ_F	discount rate	0.06	output growth rate*	0.03
δ	birth/death rate	0.02	life expectancy	50
ν_F	population share	0.01	financier+distressed leverage*	5
ν_A	population share	0.09	housing consumption share*	0.22
ν_B	population share	0.85	aggregate Sharpe ratio*	0.20
ν_D	population share	0.05	$\nu_F + \nu_D + \nu_A + \nu_B = 1$	
Panel C: Financing				
ϕ_A	liability-asset ratio	0.4	aggregate housing LTV	0.4
ϕ_B	liability-asset ratio	0.26	household credit share*	0.42
Δ_A	diversification	0.59	1990 mortgage diversification	0.59
Δ_B	diversification	0.90	syndicated bank loan spread*	0.015
$\bar{\lambda}$	maximal leverage	14	binding constraint probability*	0.03
χ	distress cost	0.03	maximal funding cost increase*	$0.02 - 0.03$

Table 2: Parameter values and targets. Housing moments are taken from Piazzesi and Schneider (2016) and Davis and Van Nieuwerburgh (2015). Idiosyncratic stock volatility is from Di Tella (2017). Financial leverage and crisis probability (binding constraint probability) are from He and Krishnamurthy (2014). The syndicated loan spread is from Sufi (2007). The 0.42 household credit share is the 1985 value of the series in figure 1. In the model, household credit share is computed as $(\kappa + 0.1)\phi_A/((\kappa + 0.1)\phi_A + (1 - \kappa)\phi_B)$ to account for the approximately 1/3 share of household finance that was non-mortgage finance in the 1980s ($\kappa = 0.25$ with a capital-housing ratio of 3). The maximal funding cost increase is taken from the $> 2\%$ estimate of Fleckenstein and Longstaff (2018). Targets with stars (*) are only matched approximately.

Into the model, I feed in a series for $\Delta_{A,t}$ that approximately matches figure 13. I assume $\Delta_{A,t} = 0.59$ for $t \in [1980, 1990]$. Then, $\Delta_{A,t}$ increases linearly from 1990 until 2006, where $\Delta_{A,2006} = 0.91$. The resulting series is depicted in the left panel of figure 15.

To extract the two-dimensional Brownian shocks (Z_t^A, Z_t^B), I approximately match two model-implied series to the data, from 1980 to 2015: (a) log GDP and (b) the log household credit share

(each with 50% weights) i.e.,

$$\log(\text{GDP}) = \log\Big(K[\kappa G_A + (1-\kappa)G_B - (1-x)\eta(\lambda_D^A + \lambda_D^B)]\Big)$$
$$\log(\text{household credit share}) = \log\Big(\frac{(\kappa+0.1)\phi_A}{(\kappa+0.1)\phi_A + (1-\kappa)\phi_B}\Big).$$

The 0.1 wedge in the household credit share is to account for the fact that mortgage credit only accounts for approximately $2/3$ of household credit. The extracted shock series are depicted in the right panel of figure 15. In all figures, "model" refers to the model with shocks to both Δ_A and (Z^A, Z^B). "Counterfactual" refers to the model with the same shocks to (Z^A, Z^B) but assumes Δ_A constant. In both the model with diversification shocks and the counterfactual exercise, I use the binomial approximation to Brownian motion, i.e., $dZ = \pm\sqrt{dt}$. Both exercises are initialized with the state variables (α, η, x) at their stationary averages.

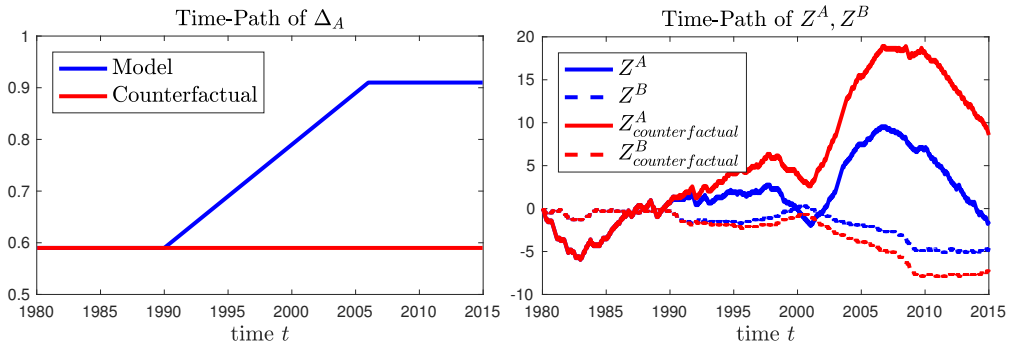

Figure 15: Shocks. Parameters are in table 2.

The data series used for shock extraction, and their model counterparts, are depicted in figure 16. Both models roughly match aggregate output and household credit. However, referring back to figure 15, we see the counterfactual model's shocks are larger, thus less likely from an ex-ante perspective. For example, the probability of observing $|Z^A - Z^B|$ increase by at least 22 over 10 years, as the counterfactual shocks display in the period 2000-2010, is equal to 8.6×10^{-7}. By contrast, the model with diversification improvements implies $|Z^A - Z^B|$ increases by 12 over the same period, which has probability 7.3×10^{-3}, four orders of magnitude larger.[37] The large counterfactual expansion of $|Z^A - Z^B|$ from 2000-2010 helps explain the large growth in household credit, which diversification improvements naturally generate through the reallocation effect. Repeating this analysis for 2005-2010 reveals probabilities of 0.0057 (counterfactual) and 0.1714 (model).

Without diversification shocks, financier leverage does not build up, and leverage constraints are no concern. Figure 17 compares the significant model-generated financial distress in the 2008-10 period, compared to a complete lack of distress in the counterfactual model. Referring back to figure 16, notice the distress period generates an acute housing bust, even moreso than in the data.

[37] These probabilities are calculated using the fact that $M := |Z^A - Z^B|/\sqrt{2}$ is a *reflected Brownian motion*. The probability distribution is given by $\mathbb{P}(M_t \geq m) = 2\Phi(-m/\sqrt{t})$, where $\Phi(\cdot)$ denotes the standard normal cdf.

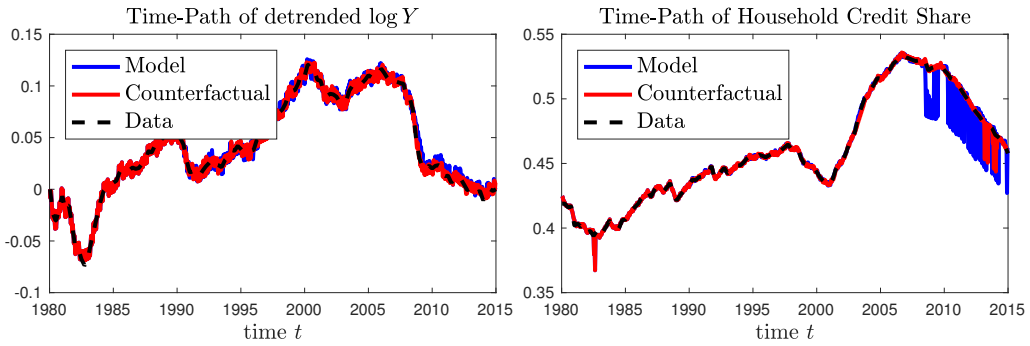

Figure 16: Log GDP (left panel) and household credit share (right panel). Parameters are in table 2.

Before the distress, during the boom years, diversification improvements reduce sector A spreads, as in the data. This force operates somewhat independently of sector B spreads, as shown in figure 18. But as distress arises, s_A and s_B spike about 2.5% and move together thereafter, nearly one-for-one. Spreads move more closely together in busts, because their behavior is determined by financiers' health issues, rather than sectoral concerns. In this period, spreads reflect almost exactly the behavior of the shadow-funding cost ζ, which approximately matches the 2+% estimate of Fleckenstein and Longstaff (2018).

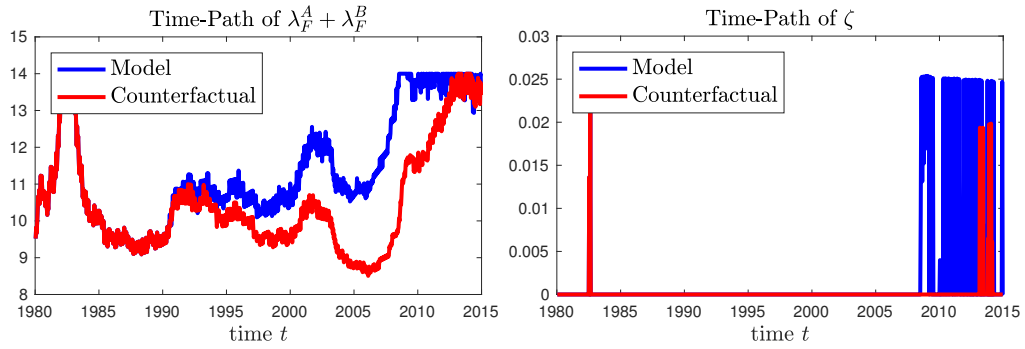

Figure 17: Financier leverage (left panel) and shadow funding cost (right panel). Parameters are in table 2.

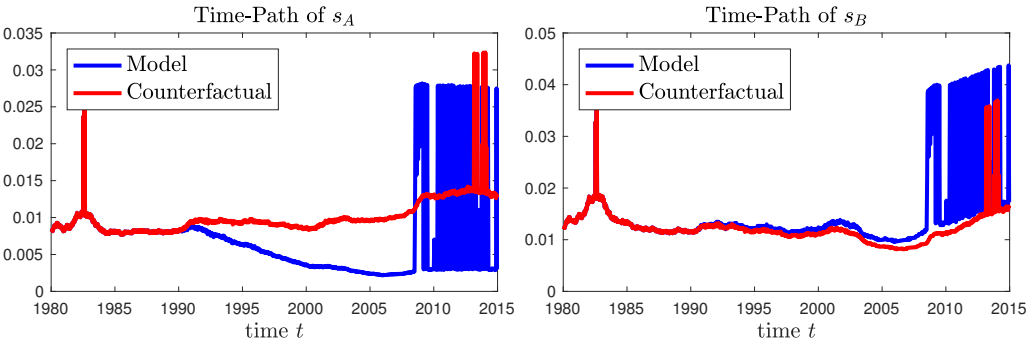

Figure 18: Lending spreads. Parameters are in table 2.

Quality Gradient

The previous calibration uses parameters (e.g., LTV ϕ_A and idiosyncratic risk $\hat{\sigma}_A$) relevant for the *average* household borrower. In this section, I ask how much stronger are the effects of diversification improvements if the model is instead calibrated to *marginal* household borrowers, who tend to be riskier (higher $\hat{\sigma}_A$) and more external-finance dependent (higher ϕ_A).

This question is interesting for two reasons. First, diversification improvements in the 1990s and especially 2000s were likely larger for lower-quality borrowers. For example, securitization of non-conforming loans (private-label MBS containing subprime, alt-A, and jumbo loans) increased dramatically in the 2000s, even relative to conforming loans (see Appendix D.2). By contrast, figure 13 only shows the average increase in diversification over all mortgage originations, which hides any potential heterogeneity.

Second, for a given increase in diversification, the effects are likely to be stronger for these marginal borrowers. Theoretically, suppose sector A has a greater amount of idiosyncratic risk ($\hat{\sigma}_A > \hat{\sigma}_B$) and borrows against a greater fraction of its asset purchases ($\phi_A > \phi_B$).[38] Then, a diversification boom in sector A tends to produce larger reallocation and leverage effects than a diversification boom in sector B. The intuition for this statement comes from idiosyncratic risk prices, e.g., $\hat{\pi}_{F\to A}$. Larger $\hat{\sigma}_A$ or larger ϕ_A imply a larger reduction in risk prices from diversification improvements, i.e.,

$$\frac{d^2\hat{\pi}_{F\to A}}{d\Delta_A d\hat{\sigma}_A} < 0 \quad \text{and} \quad \frac{d^2\hat{\pi}_{F\to A}}{d\Delta_A d\phi_A} < 0, \quad \text{holding fixed } \kappa.$$

This is why improved diversification of lower-quality borrowers' idiosyncratic risks might lead to a large cycle, and it helps reconcile the timing of the 2000s housing boom with the timing of the 2000s private-label MBS boom, rather than an earlier increase in diversification of conforming mortgages.[39]

In the quantitative experiment, I increase ϕ_A from 0.40 to 0.60 and $\hat{\sigma}_A$ from 0.11 to 0.20. To match the targets in table 2, I also adjust the following parameters: $G_A = 0.05$; $\rho_F = 0.155$; $\nu_A = 0.50$; $\nu_B = 0.44$; $\phi_B = 0.389$; and $\bar{\lambda} = 27$.[40] The procedure for extracting (Z^A, Z^B) is the same as before.

Figures 19 and 20 show financier leverage, financiers' shadow-funding cost, and both sectoral spreads. Under this marginal borrower calibration, the boom coincides with a sustained decline in s_A, on the order of 2%, in line with the drop in spreads documented in Justiniano et al. (2017), and twice as large as the baseline parameterization. At the same time, diversification-induced leveraging is massive: financiers lever up from 12 to 27 between 1990 and 2007, which is a similar order of magnitude to the broker-dealer leverage increase documented in figure 1. These strengthened

[38] One microfoundation for why lower-quality insiders might have higher issuance is to introduce asymmetric information about insider types. In a standard signalling equilibrium, higher-quality types must retain a greater share of risk in order to separate themselves from low-quality types. When interpreting ϕ_z as a borrowing constraint, it is not clear why lower-quality borrowers would have looser borrowing constraints. But when interpreting ϕ_z as a reduced-form for borrowing demand, it becomes reasonable to assume lower-quality types (who are typically poorer) will have higher ϕ_z.

[39] See Mian and Sufi (2018) for evidence that the private-label MBS boom caused a large housing boom.

[40] This calibration, primarily due to the high value of $\bar{\lambda}$, reflects an approximately 0.1% probability of leverage constraints binding, under $\Delta_A = 0.59$. This is significantly lower than the 3% target from table 2, but this target is intentionally underestimated, because diversification improvements are much stronger under this calibration, as figures 19 and 20 illustrate.

effects can be attributed to the discussion above: diversification improvements have a larger impact on riskier (higher $\hat{\sigma}_A$) and more external-finance-dependent (higher ϕ_A) borrowers.

In 2007, leverage constraints are hit, and a financial crisis occurs, upon which spreads in both sectors jump by about $\chi = 3\%$. Under this calibration, spreads remain elevated for longer than in the baseline calibration. Finally, the counterfactual without diversification improvements generates no financial crisis at all.

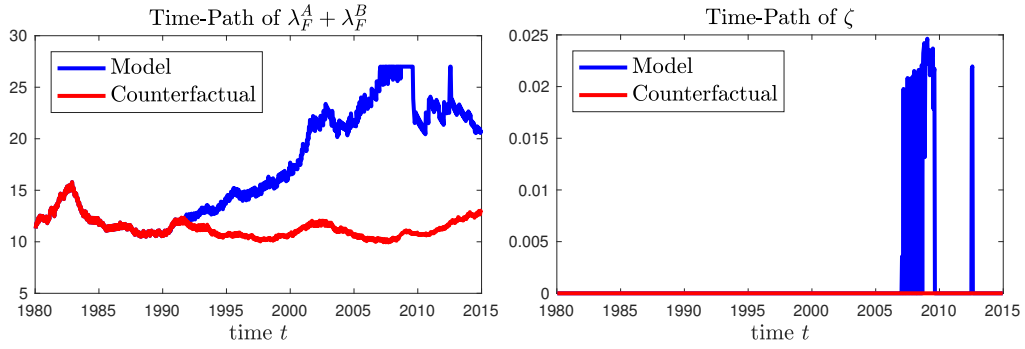

Figure 19: Financier leverage (left panel) and shadow funding cost (right panel). Parameters are in table 2, with the exception of the modifications in the text.

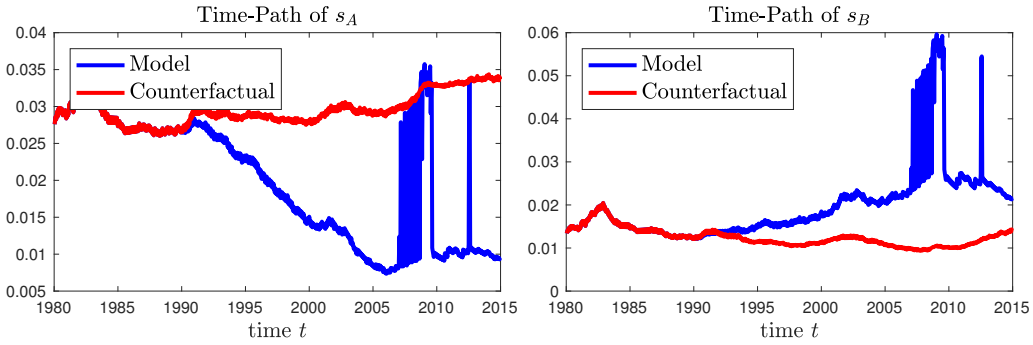

Figure 20: Lending spreads. Parameters are in table 2, with the exception of the modifications in the text.

6 Conclusion

I show that a sector-specific diversification improvement can generate a sectoral boom followed by a broader bust. The recent US housing cycle appears to be a good example, with evidence of rising diversification in mortgage markets, followed by high house prices and rising intermediary leverage. The key to these dynamics is that the diversification shock be both asset- and agent-specific. Future research on this subject can go in several directions, and I briefly mention a few.

A weakness of my framework is the exogeneity of diversification. In reality, marketing of securitized products, creation of robust banking networks, and even financial deregulations are endoge-

nous. Furthermore, there are likely to be linkages between diversification and other financial variables, such as credit standards and collateral constraints. Understanding these dynamics requires more detailed theoretical analysis of the interplay between economic conditions and diversification.

My quantitative application focuses on the recent housing boom, because diversification plausibly increased more in mortgages than business credit. But as Mian, Sufi and Verner (2017a) show, household credit is generally a better predictor of future recessions than non-household credit. What is special about housing, as it pertains to boom-bust cycles? Future work should go beyond exogenous housing-specific shocks and try to understand why the effects of neutral-seeming financial shocks (such as a global savings glut) might be stronger in housing markets.

In my model, consistent with Mian et al. (2017a), growing household debt does predict low growth and macroeconomic instability. However, the channel operates entirely through financier leverage, as opposed to household distress and defaults. Mian, Sufi and Verner (2017b) show the 1980s housing boom, like the 2000s boom, was accompanied both by a significant amount of bank failures and household defaults. Quantitatively, are busts more sensitive to weak household balance sheets or weak intermediary balance sheets?

Finally, we can extend the framework of this paper to study an asymmetric equilibrium with geographic heterogeneity, motivated by the large heterogeneity in timing and scale of regional housing cycles. In such a model, how large are the spillovers across regions from localized shocks? How do these spillovers depend on the degree of market integration? Studying such an asymmetric equilibrium is challenging, but Khorrami (2018) provides one step in this direction.

A Model Proofs and Derivations

A.1 Optimal Choices

In this section, I apply the convex duality approach of Cvitanić and Karatzas (1992) to solve agents' portfolio problems. This is a generalization of the martingale approach of Karatzas et al. (1987) and Cox and Huang (1989) to allow for portfolio constraints. I solve a slightly more general portfolio problem that nests the problems of insiders, financiers, and distressed investors.

Problem Setup

In general, all agents have a version of the following budget constraint:

$$dn_t = n_t[\mu_t^n dt + \sigma_t^n dZ_t + \hat{\sigma}_t^n d\hat{Z}_t], \quad n_0 > 0, \tag{43}$$

where Z and $\hat{Z}$ are two independent standard Brownian motions of dimensions D and M (Z is the vector of aggregate shocks), and

$$\begin{aligned}\mu_t^n &= r_t - \frac{c_t}{n_t} + \theta_t \pi_t + \lambda_t(a_t - r_t \mathbf{1})\\ \sigma_t^n &= \theta_t + \lambda_t b_t\\ \hat{\sigma}_t^n &= \lambda_t \hat{b}_t.\end{aligned}$$

By appropriate definition of the variables $a_t \in \mathbb{R}^M$, $b_t \in \mathbb{R}^M \times \mathbb{R}^D$, and $\hat{b}_t \in \mathbb{R}^M \times \mathbb{R}^M$, equation (43) can replicate insiders', financiers', or distressed investors' net worth evolutions. For example, diversification Δ_A, Δ_B is accounted for by putting $\hat{b}_{F,t} = \text{diag}(\hat{b}_{F,A,t}, \hat{b}_{F,B,t})$ with $\hat{b}_{F,z,t} = (1-\Delta_z)\hat{\sigma}_z$ and considering $\hat{Z}_t = (W_{i,t}^{A,\Delta}, W_{i,t}^{B,\Delta})'$, with $W_{i,t}^{z,\Delta} := (1-\Delta_z)^{-1}\Delta_z^{-1}\int_i^{i+\Delta_z} W_{j,t}^z dj$ a Brownian motion by Lemma 2.2.

In addition, we have the following portfolio constraints:

$$\lambda_t \in \Lambda \quad \text{and} \quad \theta_t \in \Theta \tag{44}$$

for $\Lambda := \{\lambda : \lambda \geq 0, \lambda \mathbf{1} \leq \bar{\lambda}\} \subset \mathbb{R}^M$ and either $\Theta := \mathbb{R}^D$ or $\Theta := \{0\}^D$. For insiders and distressed investors, $\bar{\lambda} = +\infty$. For insiders, in the case they cannot trade any aggregate risks, $\Theta = \{0\}^D$. When they can trade aggregate risks, $\Theta = \mathbb{R}^D$. This is always the case for financiers and distressed investors.

We are now in position to state agents' optimization problems, which are all sub-cases of the following. For $\mathcal{U}$ the logarithmic utility function defined in (1), agents solve

$$\mathcal{U}_t^* := \sup_{n,c,\lambda,\theta} \mathcal{U}_t \tag{45}$$

subject to (43), (44), and $n_t \geq 0$.

The heuristic derivation of optimal controls is as follows. The necessary technical arguments are presented in Cvitanić and Karatzas (1992). For any convex set $\mathcal{A}$, define the penalty function

$$\varphi_{\mathcal{A}}(x) := \begin{cases} 0, & \text{if} \quad x \in \mathcal{A};\\ -\infty, & \text{if} \quad x \notin \mathcal{A}.\end{cases}$$

Augment the wealth dynamics by φ_Λ and φ_Θ to account for the portfolio constraints (44):

$$dn_t = n_t[\mu_t^n dt + \varphi_\Lambda(\lambda_t)dt + \varphi_\Theta(\theta_t)dt + \sigma_t^n dZ_t + \hat{\sigma}_t^n d\hat{Z}_t].$$

Introduce an Itô process (which will represent the state-price density or Lagrange multiplier process):

$$d\xi_t = -\xi_t \left[\alpha_t dt + \beta_t \cdot dZ_t + \hat{\beta}_t \cdot d\hat{Z}_t\right]. \tag{46}$$

Itô's formula implies

$$\begin{aligned}\xi_\tau n_\tau = \xi_0 n_0 &+ \int_0^\tau \xi_t n_t\Big(- \alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t\Big)dt \\ &+ \int_0^\tau \xi_t n_t\Big(- \beta_t' + \sigma_t^n\Big)dZ_t + \int_0^\tau \xi_t n_t\Big(- \hat{\beta}_t' + \hat{\sigma}_t^n\Big)d\hat{Z}_t.\end{aligned} \tag{47}$$

Now, we want to take expectations to eliminate the stochastic integrals, and then to take $\tau \to +\infty$. Doing this requires a series of technical arguments.

First, τ may be a stopping time rather than a deterministic time. In particular, the equilibrium of the model will imply, in principal, that $-r_t$ and π_t can be arbitrarily large, so I localize the integral with $\tau \equiv \tau_L := T \wedge \tau_L^{-r} \wedge \tau_L^{-\alpha} \wedge \tau_L^{\pi} \wedge \tau_L^{\beta} \wedge \tau_L^{\hat{\beta}}$, where $T > 0$ is deterministic and for any process x we have defined $\tau_L^x := \inf\{t \geq 0 : x_t \geq L\}$ for some $L > 0$. However, equilibrium will have the property that $\lim_{L\to\infty} \tau_L = T$ almost-surely, because the probability of large $-r_t$ or π_t vanishes (this can be verified ex-post using the equilibrium state dynamics from Proposition 3.1). Consequently, we may take expectations, followed by the limit $L \to +\infty$, to obtain

$$\mathbb{E}[\xi_T n_T] = \xi_0 n_0 + \mathbb{E}\int_0^T \xi_t n_t\Big(- \alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t\Big)dt,$$

where τ is replaced with T inside the expectations by the dominated convergence theorem, which holds as long as $\lambda \in \Lambda$ and $\theta \in \Theta$. Indeed, the coefficients of ξ and n are uniformly bounded up to time T.

Because maximization will imply a transversality condition on discounted wealth, assume $\lim_{T\to\infty} \xi_T n_T = 0$ almost-surely. The transversality condition will have to be verified in equilibrium. If it holds, then we may apply appropriate convergence theorems to take $T \to +\infty$. Indeed, we may split $\xi_t n_t(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t)$ into positive and negative parts and apply the monotone convergence theorem separately to the integrals of these parts. Furthermore, by ignoring the negative part, we have $\mathbb{E}[\xi_T n_T] \leq \xi_0 n_0 + \mathbb{E}\int_0^\infty \xi_t n_t(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t)^+ dt$. This upper bound implies we may apply the dominated convergence theorem to take $\lim_{T\to\infty} \mathbb{E}[\xi_T n_T] = 0$. The result of taking these limits is

$$0 = \xi_0 n_0 + \mathbb{E}\int_0^\infty \xi_t n_t\Big(- \alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \cdot \beta_t - \hat{\sigma}_t^n \cdot \hat{\beta}_t\Big)dt. \tag{48}$$

The "static" budget constraint (48) is an implication of the dynamic wealth constraint (47), which means that the result of the unconstrained problem

$$\sup_{n\geq 0, c, \lambda, \theta} \mathbb{E}\Big[\int_0^\infty \Big(\rho e^{-\rho t}\log c_t + \xi_t n_t\Big(- \alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n \beta_t - \hat{\sigma}_t^n \hat{\beta}_t\Big)\Big)dt + \xi_0 n_0\Big] \tag{49}$$

is technically an upper bound on the maximized constrained objective (64). The point of Cvitanić and Karatzas (1992), Theorem 10.1, is to show that by minimizing over the process ξ in (46), one can obtain the value of

the maximized constrained objective, i.e.,

$$\mathcal{U}_0^* = \inf_{\xi} \sup_{n\geq 0,c,\lambda,\theta} \mathbb{E}\Big[\int_0^\infty \Big(e^{-\rho t}\log c_t + \xi_t n_t\Big(-\alpha_t + \mu_t^n + \varphi_\Lambda(\lambda_t) + \varphi_\Theta(\theta_t) - \sigma_t^n\beta_t - \hat{\sigma}_t^n\hat{\beta}_t\Big)\Big)dt + \xi_0 n_0\Big]. \quad (50)$$

Furthermore, the order of minimization and maximization may be exchanged. With this equivalence, optimal policies can be found from the unconstrained problem (49) for *some* process ξ that is suitably minimal.

Solving the Problem

First, we solve the maximization problem. The first-order condition with respect to c is typical:

$$\rho e^{-\rho t}\frac{1}{c_t} = \xi_t. \quad (51)$$

To solve for optimal portfolios, introduce the "slackness" processes

$$\nu := b\beta + \hat{b}\hat{\beta} + r\mathbf{1} - a \quad (52)$$

$$\omega := \beta - \pi. \quad (53)$$

Maximizing over (λ,θ) are thus equivalent to maximizing $\varphi_\Lambda(\lambda) - \lambda\nu$ and $\varphi_\Theta(\theta) - \theta\omega$. With that in mind, for any convex set $\mathcal{A}$ define the convex support function $\tilde{\varphi}_\mathcal{A}(x) := \sup_y \varphi_\mathcal{A}(y) - yx = \sup_{y\in\mathcal{A}}(-yx)$. Defining $\bar{\nu} := \min(\nu)$, these conjugate functions are given by (regardless of whether $\Theta = \mathbb{R}^D$ or $\Theta = \{0\}^D$)

$$\tilde{\varphi}_\Lambda(\nu) = \bar{\lambda}\max(0,-\bar{\nu}) \quad (54)$$

$$\tilde{\varphi}_\Theta(\omega) = 0. \quad (55)$$

Note that, when $\Theta = \mathbb{R}^D$, it must be the case that $\omega_t \equiv 0$. Finally, substituting (51), (54), and (55) back into the objective (49), we have

$$\sup_{n\geq 0}\mathbb{E}\Big[\int_0^\infty \Big(\rho e^{-\rho t}[-1 + \log\rho - \rho t - \log\xi_t] + \xi_t n_t[-\alpha_t + r_t + \bar{\lambda}\max(0,-\bar{\nu}_t)]\Big)dt + \xi_0 n_0\Big].$$

Assuming $n_t > 0$ for all t (which can be verified ex-post by the optimal wealth dynamics), maximizing over n implies that

$$-\alpha_t + r_t + \max(0,-\bar{\lambda}\bar{\nu}_t) = 0. \quad (56)$$

Next, minimizing over ξ in (50) amounts to minimizing over (ν,ω) and the initial value ξ_0. This is because the coefficients of ξ depend on market prices and the process (ν,ω), as seen in the necessary conditions (52), (53), and (56). To emphasize this dependence, write $\xi^{\nu,\omega}$, $\alpha^{\nu,\omega}$, $\beta^{\nu,\omega}$, and $\hat{\beta}^{\nu,\omega}$ for the Lagrange multiplier process and its coefficient processes. In particular, note that

$$\xi_t^{\nu,\omega} = \xi_0\exp\Big\{-\int_0^t\Big(\alpha_s^{\nu,\omega} + \frac{1}{2}\|\beta_s^{\nu,\omega}\|^2 + \frac{1}{2}\|\hat{\beta}_s^{\nu,\omega}\|^2\Big)ds - \int_0^t \beta_s^{\nu,\omega}\cdot dZ_s - \int_0^t \hat{\beta}_s^{\nu,\omega}\cdot d\hat{Z}_s\Big\} \quad (57)$$

$$\alpha_t^{\nu,\omega} = r_t + \bar{\lambda}\max(0,-\bar{\nu}_t) \quad (58)$$

$$\beta_t^{\nu,\omega} = \omega_t + \pi_t \quad (59)$$

$$\hat{\beta}_t^{\nu,\omega} = \hat{b}_t^{-1}[a_t - r_t\mathbf{1} - b_t(\omega_t + \pi_t) + \nu_t] \quad (60)$$

We are led to solve the dual problem

$$\inf_{\nu,\omega,\xi_0} -\mathbb{E}\Big[\int_0^\infty \rho e^{-\rho t}\log \xi_t^{\nu,\omega} dt - \xi_0 n_0\Big], \tag{61}$$

subject to (57), (58), (59), (60), and additionally $\omega_t = 0$ if we set $\Theta = \mathbb{R}^D$.

Substituting $\xi^{\nu,\omega}$ into the objective (61), we immediately solve for the initial condition and find that

$$\xi_0 = \frac{1}{n_0}. \tag{62}$$

Then, assuming we can perform appropriate localizations on the stochastic integrals in (57) as before, the processes (ν,ω) are determined from solving

$$\inf_{\nu,\omega}\mathbb{E}\Big[\int_0^\infty \rho e^{-\rho t}\int_0^t \Big(r_s + \bar\lambda\max(0,-\bar\nu_s) + \frac{1}{2}\|\omega_s+\pi_s\|^2 + \frac{1}{2}\|\hat b_s^{-1}(a_s - r_s\mathbf{1} - b_s(\omega_s+\pi_s)+\nu_s)\|^2\Big)dsdt\Big].$$

Crucially, notice that the minimization can be taken pointwise, i.e.,

$$\omega_t = \arg\min_{x\in\mathbb{R}^D}\Big\{\frac{1}{2}\|x+\pi_t\|^2 + \frac{1}{2}\|\hat b_t^{-1}(a_t - r_t\mathbf{1} - b_t(x+\pi_t)+\nu_t)\|^2\Big\}$$

and

$$\nu_t = \arg\min_{x\in\mathbb{R}^M}\Big\{\bar\lambda\max(0,-\min(x)) + \frac{1}{2}\|\hat b_t^{-1}(a_t - r_t\mathbf{1} - b_t(\omega_t+\pi_t)+x)\|^2\Big\}.$$

These are convex problems and have unique solutions. For reference, these are the same as equation (11.4) in Cvitanić and Karatzas (1992).

Recall that $\omega_t = 0$ if $\Theta = \mathbb{R}^D$. If $\Theta = \{0\}^D$ instead, then by inspection we see that $\omega_t = -\pi_t$ is the optimal choice.

Now we solve for ν. In all of the applications in the paper, $\hat b$ is a diagonal matrix with dimension $M = 2$. To solve this problem, I specialize to this case, which simplifies the calculations. Now

$$\|\hat b^{-1}(a - r\mathbf{1} - b(\omega+\pi)+\nu)\|^2 = \sum_{i=1}^2\Big(\frac{[a]_i - r - [b(\omega+\pi)]_i + [\nu]_i}{[\hat b]_{ii}}\Big)^2,$$

where $[x]_i$ and $[y]_{ij}$ represent the ith element of the vector x and (i,j)th element of the matrix y. Define $\hat\pi_i := [\hat b]_{ii}^{-2}([a]_i - r - [b(\omega+\pi)]_i)$. Minimizing with respect to ν requires a case-by-case analysis, similar to example 14.9 in Cvitanić and Karatzas (1992):

- $\hat\pi_1 \le 0, \hat\pi_2 \le 0$.

  Optimal choice: $[\nu]_1 = -[\hat b]_{11}^2\hat\pi_1$ and $[\nu]_2 = -[\hat b]_{22}^2\hat\pi_2$.

  Rationale: If either $\hat\pi_i \le 0$, the optimal choice for $[\nu]_i$ can be made independent of $[\nu]_{-i}$ and that choice is $[\nu]_i = -[\hat b]_{ii}^2\hat\pi_i$.

- $\hat\pi_1 > 0, \hat\pi_2 \le 0$.

  Optimal choice: $[\nu]_1 = -[\hat b]_{11}^2(\hat\pi_1 - \bar\lambda)^+$ and $[\nu]_2 = -[\hat b]_{22}^2\hat\pi_2$.

  Rationale: $[\nu]_2$ can be chosen according to the previous case.

  If $\hat\pi_1 > \bar\lambda$, then the choice of $[\nu]_1 = -[\hat b]_{11}^2(\hat\pi_1 - \bar\lambda) = \bar\nu < 0$ minimizes $-\bar\lambda\bar\nu + \frac{1}{2}[\hat b]_{11}^{-2}([\hat b]_{11}^2\hat\pi_1 + \bar\nu)^2$.

  If $\hat\pi_1 \le \bar\lambda$, then it must be that $\bar\nu \ge 0$ in which case $[\nu]_1 = 0$ minimizes $\frac{1}{2}[\hat b]_{11}^{-2}([\hat b]_{11}^2\hat\pi_1 + [\nu]_1)^2$.

- $\hat{\pi}_1 \leq 0, \hat{\pi}_2 > 0$.

  Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2 \hat{\pi}_1$ and $[\nu]_2 = -[\hat{b}]_{22}^2 (\hat{\pi}_2 - \bar{\lambda})^+$.

  Rationale: This case is symmetrical to the previous one.

- $\hat{\pi}_1 > 0, \hat{\pi}_2 > 0$ and $\hat{\pi}_1 + \hat{\pi}_2 \leq \bar{\lambda}$.

  Optimal choice: $[\nu]_1 = 0$ and $[\nu]_2 = 0$.

  Rationale: Choosing either (or both) $[\nu]_1, [\nu]_2 < 0$ is not feasible because first-order optimality cannot be satisfied. Moreover, choosing $[\nu]_1, [\nu]_2 > 0$ is not optimal, leaving the zero solution.

- $\hat{\pi}_1 > 0, \hat{\pi}_2 > 0$ and $\hat{\pi}_1 + \hat{\pi}_2 > \bar{\lambda}$.

  Here, it must be the case that $\nu \leq 0$ with at least one of $[\nu]_1, [\nu]_2$ strictly negative. Consider the three sub-cases $[\nu]_1 < [\nu]_2 \leq 0$, $[\nu]_2 < [\nu]_1 \leq 0$, and $[\nu]_1 = [\nu]_2 < 0$. In the first case, the optimal choices are $[\nu]_1 = [\hat{b}]_{11}^2 [\bar{\lambda} - \hat{\pi}_1]$ and $[\nu]_2 = -[\hat{b}]_{22}^2 \hat{\pi}_2$, and these two must be ordered as anticipated. The second case is symmetrical. The third case with $[\nu]_1 = [\nu]_2 = \bar{\nu}$ has the optimality condition $\bar{\nu} = ([\hat{b}]_{11}^{-2} + [\hat{b}]_{22}^{-2})^{-1} [\bar{\lambda} - \hat{\pi}_1 - \hat{\pi}_2]$. Thus, we have the three corresponding sub-cases.

  * $\hat{\pi}_1 - [\hat{b}]_{11}^{-2} [\hat{b}]_{22}^2 \hat{\pi}_2 \geq \bar{\lambda}$.
    Optimal choice: $[\nu]_1 = [\hat{b}]_{11}^2 [\bar{\lambda} - \hat{\pi}_1]$ and $[\nu]_2 = -[\hat{b}]_{22}^2 \hat{\pi}_2$.
  * $\hat{\pi}_2 - [\hat{b}]_{22}^{-2} [\hat{b}]_{11}^2 \hat{\pi}_1 \geq \bar{\lambda}$.
    Optimal choice: $[\nu]_1 = -[\hat{b}]_{11}^2 \hat{\pi}_1$ and $[\nu]_2 = [\hat{b}]_{22}^2 [\bar{\lambda} - \hat{\pi}_2]$.
  * $\bar{\lambda} > \max\left\{\hat{\pi}_1 - [\hat{b}]_{11}^{-2} [\hat{b}]_{22}^2 \hat{\pi}_2, \hat{\pi}_2 - [\hat{b}]_{22}^{-2} [\hat{b}]_{11}^2 \hat{\pi}_1\right\}$.
    Optimal choice: $[\nu]_1 = [\nu]_2 = ([\hat{b}]_{11}^{-2} + [\hat{b}]_{22}^{-2})^{-1} [\bar{\lambda} - \hat{\pi}_1 - \hat{\pi}_2]$.

Consumption and Portfolios

Now, we use the solution of the dual problem to determine optimal policies. First, substitute the optimality conditions (52), (53), (54), (55), and (56) into the time-t version of the static budget constraint (48), which shows that optimal wealth is given by

$$\xi_t^{\nu,\omega} n_t = \mathbb{E}_t \Big[\int_0^\infty \xi_{t+s}^{\nu,\omega} c_{t+s} ds \Big].$$

Using (51), we obtain the familiar log utility consumption rule $c_t = \rho n_t$. Second, substitute these optimality conditions, and (62), into the dynamic budget constraint (47) to obtain

$$0 = \int_0^T \xi_t n_t (-(\beta_t^{\nu,\omega})' + \sigma_t^n) dZ_t + \int_0^T \xi_t n_t (-(\hat{\beta}_t^{\nu,\omega})' + \hat{\sigma}_t^n) d\hat{Z}_t, \quad \text{i.e.,} \quad (\sigma_t^n)' = \beta_t^{\nu,\omega} \quad \text{and} \quad (\hat{\sigma}_t^n)' = \hat{\beta}_t^{\nu,\omega}.$$

Thus, using our explicit solution for ν in the $M = 2$ case with $\hat{b}$ a diagonal matrix, the optimal λ is determined as

$$\lambda_t = \hat{b}_t^{-1} \hat{\beta}_t^{\nu,\omega} = \hat{b}_t^{-2} (a_t - r_t \mathbf{1} - b_t (\omega_t + \pi_t) - \zeta_t)^+,$$

where $\zeta_t := -\max(0, -\bar{\nu}_t)$. This is the generalization of equations (25) and (23), which are obtained using $\omega_t = 0$ (since financiers and distressed investors can access aggregate Arrow markets without constraints) and substituting appropriate $a, b, \hat{b}$ from the model. One can verify that $\zeta_t > 0$ only when $\hat{b}_t^{-2} (a_t - r_t \mathbf{1} - b_t (\omega_t + \pi_t))^+ \cdot \mathbf{1} \geq \bar{\lambda}$. Given our formula for λ_t and this observation, this verifies the complementary slackness formula (24).

A.2 General Consumption-Portfolio Problem for Recursive Utility Agents

In some extensions, I will want to consider more general preferences than log, which requires a dynamic programming method, unlike Appendix A.1. Relatedly, to analyze mobility decisions under Assumption 2, it is important to have agents' dynamic programming equations.

Suppose agents' have recursive Duffie and Epstein (1992) utility recursions, given by

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \varphi(c_s,\mathcal{U}_s)ds\Big],$$
$$\text{where} \quad \varphi(c,\mathcal{U}) := \frac{\rho(1-\gamma)\mathcal{U}}{1-\varsigma}\Big(c^{1-\varsigma}[(1-\gamma)\mathcal{U}]^{-\frac{1-\varsigma}{1-\gamma}}-1\Big). \tag{63}$$

In (63), $\rho>0$ represents the subjective discount rate, $\gamma>0$ represents the coefficient of relative risk aversion (RRA), and $\varsigma^{-1}>0$ represents the elasticity of intertemporal substitution (EIS). Setting $\varsigma=\gamma$, these preferences reduce to von Neumann-Morgenstern preferences. Setting $\varsigma=1$, the utility aggregator function ς becomes logarithmic over the consumption bundle.[41] Then, as in Appendix A.1 all agents' portfolio problems can be written as

$$\max_{n,c,\theta,\lambda} \mathcal{U}_t \tag{64}$$

subject to (43), $n_t\geq 0$, $\lambda_t\in\Lambda$, and $\theta_t\in\Theta$ for closed, convex sets $\Lambda\subset\mathbb{R}^M$ and $\Theta\subset\mathbb{R}^D$. To simplify exposition, I assume $\Lambda=\{\lambda:\lambda\geq 0,\lambda\mathbf{1}\leq\bar{\lambda}\}$ as in (44) and $\Theta=(\underline{\theta}_1,\bar{\theta}_1)\times\cdots\times(\underline{\theta}_D,\bar{\theta}_D)$ for $\underline{\theta}:=(\underline{\theta}_1,\ldots,\underline{\theta}_D)\in\mathbb{R}_-^D\cup\{-\infty\}^D$ and $\bar{\theta}:=(\bar{\theta}_1,\ldots,\bar{\theta}_D)\in\mathbb{R}_+^D\cup\{-\infty\}^D$. This assumption on Θ generalizes (44). All agents except financiers have $\bar{\lambda}=+\infty$. Whenever agents can freely trade aggregate risk in Arrow markets (e.g., both financiers and distressed investors can always do this), we have $\underline{\theta}=\{-\infty\}^D$ and $\bar{\theta}=\{+\infty\}^D$. When agents cannot trade at all in these markets (e.g., in the model of Section 3, insiders cannot trade), we have $\underline{\theta}=\bar{\theta}=\{0\}^D$.

To solve (64), we first use its scaling properties to simplify the problem. Given the homotheticity of preferences combined with the linearity of wealth evolution, value functions take the form

$$\mathcal{U}_t=\frac{(n_t\xi_t)^{1-\gamma}}{1-\gamma},$$

where

$$d\xi_t=\xi_t\Big[\mu_t^\xi dt+\sigma_t^\xi dZ_t\Big]. \tag{65}$$

The process ξ_t represents the investment opportunity set of the agent and responds only to the aggregate shock Z, due to the free mobility condition, Assumption 2.[42]

Then, the HJB equation of such an agent is given by

$$0=\max_{c,\lambda\in\Lambda,\theta\in\Theta}\Big\{\varphi(c,\mathcal{U})+n\mu^n\partial_n\mathcal{U}+\frac{1}{2}n^2[\|\sigma^n\|^2+\|\hat{\sigma}^n\|^2]\partial_{nn}\mathcal{U}+\xi\mu^\xi\partial_\xi\mathcal{U}+\frac{1}{2}\xi^2\|\sigma^\xi\|^2\partial_{\xi\xi}\mathcal{U}+n\xi\sigma^n(\sigma^\xi)'\partial_{n\xi}\mathcal{U}\Big\}.$$

[41] By taking the limit $\varsigma\to 1$ with L'Hôpital's rule, the aggregator becomes

$$\varphi(c,\mathcal{U})=\rho(1-\gamma)\mathcal{U}\Big[\log(c)-\frac{1}{1-\gamma}\log[(1-\gamma)\mathcal{U}]\Big].$$

[42] Verifying this equilibrium property is straightforward. Indeed, if ξ_t were affected by idiosyncratic shocks W, then different locations would have different levels of ξ_t. Free mobility implies agents would immediately migrate to locations with higher levels of ξ_t and attain a higher value function, which is a contradiction.

Substituting the form of $\mathcal{U}$ and its derivatives, then dividing the entire HJB equation by the positive quantity $(n\xi)^{1-\gamma}$, we obtain

$$0 = \max_{c,\lambda\in\Lambda,\theta\in\Theta} \left\{ \rho \frac{(\frac{c}{n\xi})^{1-\varsigma} - 1}{1-\varsigma} + \mu^n - \frac{\gamma}{2}[\|\sigma^n\|^2 + \|\hat{\sigma}^n\|^2] + \mu^\xi - \frac{\gamma}{2}\|\sigma^\xi\|^2 + (1-\gamma)\sigma^n(\sigma^\xi)' \right\},$$

First-order optimality for this agent implies for consumption:

$$c_t = \rho^{1/\varsigma} \xi_t^{1-1/\varsigma} n_t \tag{66}$$

Optimal portfolios must satisfy the following complementary slackness conditions:

$$0 = \min\left\{\lambda', -a + (r+\zeta)\mathbf{1} + \gamma b(\sigma^n)' + (\gamma-1)b(\sigma^\xi)' + \gamma\hat{b}(\hat{\sigma}^n)'\right\} \tag{67}$$

$$0 = \min\left\{\zeta, \bar{\lambda} - \lambda\mathbf{1}\right\} \tag{68}$$

$$0 = \max\left\{\theta' - \bar{\theta}', \min\left\{\theta' - \underline{\theta}', -\pi + \gamma(\sigma^n)' + (\gamma-1)(\sigma^\xi)'\right\}\right\}. \tag{69}$$

Plugging these choices back into the HJB equation, we obtain the following:

$$0 = \rho \frac{\varsigma(\xi/\rho)^{\frac{\varsigma-1}{\varsigma}} - 1}{1-\varsigma} + r + \bar{\lambda}\zeta + \theta\left[\pi - \gamma(\sigma^n)' - (\gamma-1)(\sigma^\xi)'\right] + \frac{\gamma}{2}\left[\|\sigma^n\|^2 + \|\hat{\sigma}^n\|^2\right] + \mu^\xi - \frac{\gamma}{2}\|\sigma^\xi\|^2, \tag{70}$$

where θ and ζ are determined using conditions (67)-(69). Note that $\rho\frac{\varsigma(\xi/\rho)^{1-1/\varsigma}-\rho}{1-\varsigma} \to \rho(\log(\rho/\xi) - 1)$ as $\varsigma \to 1$. Because ξ will be a function of aggregate state variables in a Markovian equilibrium, μ^ξ and σ^ξ may be determined in terms of the derivatives of ξ by Itô's formula. Thus, (70) is a differential equation for ξ. In principle, one could develop an infinite-horizon extension of the "verification"-type arguments of Schroder and Skiadas (2003), which nests the choice problem above aside from the finite horizon. This would show that solving equation (70) is sufficient for optimality of the choices outlined above.

Appendix A.1 proves the convex duality approach yields exactly these optimality conditions for log utility. There is no need to solve (70) in this case.

A.3 Derivation of Equilibrium

Proof of Lemma 2.3. I prove a more general version of Lemma 2.3, which applies to the model of Section 3 and allows for general Epstein-Zin preferences with common risk aversion γ and common intertemporal substitution elasticity ς^{-1} as in Appendix A.2. To accommodate the presence of distressed investors, an additional assumption is required, that $n^D_{i,t}/(k^A_{i,t} + k^B_{i,t})$ is independent of i. Lemma 2.3 can be deduced by setting $\gamma = \varsigma = 1$, $\rho_F = \rho$, $\delta = 0$, $\bar{\lambda} = +\infty$, and either $\chi = +\infty$ (so that $x_t \to 1$ as $t \to \infty$) or $\chi = 0$ (so that financiers and distressed investors are equivalent, such that x is irrelevant in equilibrium).

First, the capital return distribution from (5)-(6) is location invariant ("LI" for short in this proof). This implies that all insiders' consumption and portfolio choices are LI, see Appendix A.1 and A.2 (for non-logarithmic utility). Consequently, $\hat{\sigma}^n_A$ and $\hat{\sigma}^n_B$ must be LI. Since we know $\hat{\sigma}^n_A = (k^A/n^A)\hat{\sigma}_A$ and $\hat{\sigma}^n_B = (k^B/n^B)\hat{\sigma}_B$ are LI, we have k^A/n^A and k^B/n^B are LI. Under the stated assumption that $n^A/(n^A + n^B)$ is LI, it must be that k^A/k^B is LI.

Second, if insiders cannot trade aggregate risk, $\sigma^n_A = (\hat{\sigma}^n_A/\hat{\sigma}_A)\sigma_A$ and $\sigma^n_B = (\hat{\sigma}^n_B/\hat{\sigma}_B)\sigma_B$ are both LI by extension. If insiders can trade aggregate risk, σ^n_A and σ^n_B must be LI, as the centralized Arrow market has an LI aggregate risk price π. Similarly, σ^n_F and σ^n_D must be LI, as financiers and distressed investors make

unconstrained trades in the centralized Arrow market.

Then, all agents' value processes $\xi_{A,t}$, $\xi_{B,t}$, $\xi_{F,t}$, and $\xi_{D,t}$ (defined in Appendix A.2) are automatically LI under free mobility, as explained in footnote 42. In particular, $(\mu^\xi_z, \sigma^\xi_z)$ are LI for $z \in \{A, B, F, D\}$. Using the fact that $(\sigma^n, \mu^\xi, \sigma^\xi)$ is LI in the general HJB equation (70) shows that $\bar{\lambda}\zeta + \frac{\gamma}{2}\|\hat{\sigma}^n\|^2$ must be LI.

Now, I will show that this implies $\hat{\sigma}^n_F$ and $\hat{\sigma}^n_D$ are LI. Indeed, distressed investors face no leverage constraint. Also, using the notation of Appendix A.1 and A.2, $\hat{b}^F_{i,t} = \hat{b}^D_{i,t} = \text{diag}((1-\Delta_A)\hat{\sigma}_A, (1-\Delta_B)\hat{\sigma}_B)$ is independent of i for both financiers and distressed investors. Thus, the condition that $\bar{\lambda}\zeta + \frac{\gamma}{2}\|\hat{\sigma}^n\|^2$ is LI translates into

$$\bar{\lambda}\zeta_{i,t} + \frac{\gamma}{2}[(\lambda^A_{F,i,t}(1-\Delta_A)\hat{\sigma}_A)^2 + (\lambda^B_{F,i,t}(1-\Delta_B)\hat{\sigma}_B)^2] \quad \text{is LI} \tag{71}$$

$$\text{and} \quad \frac{\gamma}{2}[(\lambda^A_{D,i,t}(1-\Delta_A)\hat{\sigma}_A)^2 + (\lambda^B_{D,i,t}(1-\Delta_B)\hat{\sigma}_B)^2] \quad \text{is LI.} \tag{72}$$

We also have equation (68) for financiers, which is

$$0 = \min\{\zeta_{i,t},\ \bar{\lambda} - \lambda^A_{F,i,t} - \lambda^B_{F,i,t}\}. \tag{73}$$

We finally have the equity market clearing condition, a modification of the condition in Definition 1 to incorporate distressed investors, which says

$$\phi_z k^z_{i,t} = \Delta_z^{-1}\int_{i-\Delta_z}^{i} [\lambda^z_{F,j,t} n^F_{j,t} + \lambda_{D,j,t} n^D_{j,t}]dj, \quad z \in \{A, B\}.$$

Define the LI quantities $\kappa_{z,t} := k^z_{i,t}/(k^A_{i,t} + k^B_{i,t})$ and $x_t\eta_t := n^F_{i,t}/(k^A_{i,t} + k^B_{i,t})$ and $(1-x_t)\eta_t := n^D_{i,t}/(k^A_{i,t} + k^B_{i,t})$. Also using the fact that $k^A_{i,t} + k^B_{i,t}$ is LI, we may re-write the equity market clearing condition as

$$\phi_z \kappa_{z,t} = \Delta_z^{-1}\int_{i-\Delta_z}^{i} \eta_t[x_t\lambda^z_{F,j,t} + (1-x_t)\lambda_{D,j,t}]dj, \quad z \in \{A, B\}. \tag{74}$$

By differentiating (74) with respect to i and combining with (71), (72), and (73), we have 5 independent expressions involving LI combinations of $(\zeta_{i,t}, \lambda^A_{F,i,t}, \lambda^B_{F,i,t}, \lambda^A_{D,i,t}, \lambda^B_{D,i,t})$, implying that each of them are LI.

Next, I will show that spreads are LI. Using the notation of Appendix A.1 and A.2, we have

$$a^F_{i,t} = \left(\Delta_A^{-1}\int_i^{i+\Delta_A} s^A_{j,t}dj,\ \Delta_B^{-1}\int_i^{i+\Delta_B} s^B_{j,t}dj\right)'$$

$$a^D_{i,t} = \left(\Delta_A^{-1}\int_i^{i+\Delta_A} (s^A_{j,t} - \chi)dj,\ \Delta_B^{-1}\int_i^{i+\Delta_B} (s^B_{j,t} - \chi)dj\right)'.$$

Thus, combining equations (67), (68), and (69), using the fact that $\Theta = \mathbb{R}^2$ is the constraint set for financiers and distressed investors, we obtain

$$\lambda^z_{F,i,t} = \frac{[\Delta_z^{-1}\int_i^{i+\Delta_z} s^z_{j,t}dj - r_t - \sigma_z\cdot\pi_t - \zeta_{i,t}]^+}{\gamma(1-\Delta_z)^2\hat{\sigma}_z^2} \quad \text{and} \quad \lambda^z_{D,i,t} = \frac{[\Delta_z^{-1}\int_i^{i+\Delta_z} s^z_{j,t}dj - r_t - \sigma_z\cdot\pi_t - \chi]^+}{\gamma(1-\Delta_z)^2\hat{\sigma}_z^2}, \quad z \in \{A, B\}.$$

Using the result that $(\zeta_{i,t}, \lambda^A_{F,i,t}, \lambda^B_{F,i,t}, \lambda^A_{D,i,t}, \lambda^B_{D,i,t})$ are LI, we have $(s^A_{i,t}, s^B_{i,t})$ are LI. This completes the verification of an LI equilibrium.

Finally, note that it is feasible to have a symmetric equilibrium where all quantities are LI. Indeed, Assumption 2 implies that $n^z_{i,t}$ can be independent of i for $z \in \{A, B, F, D\}$. The fact that capital investment is

frictionless implies that $k^A_{i,t}, k^B_{i,t}$ can be independent of i. The preceding arguments then imply that all other equilibrium objects must be LI. Such mobility of net worth is weakly optimal in a symmetric equilibrium, since it is costless, and similarly for capital since its investment is frictionless. □

Proof of Proposition 2.4. This is a special case of Proposition 3.1 with $\bar{\lambda} = +\infty$, $\chi = 0$, $\delta = 0$, and $\rho_F = \rho$, as well as unconstrained θ_A, θ_B (such that $\pi_A = \pi_B = \pi$). Substituting $\zeta = \chi = 0$ into equation (28), and then substituting the result into equation (27), we obtain equation (15). Similarly, $\chi = 0$ and $\rho_F = \rho$ in equation (26), we obtain equation (16). By time-differentiating the goods market clearing condition, we obtain equation (17). The state dynamics of (α, η) are obtained by substituting $\delta = 0$ and $\rho_F = \rho$ and $\pi_A = \pi_B = \pi$.

Existence/uniqueness follows from uniqueness of optimal choices from Appendix A.1, the explicit solution (15) for κ_t, and the explicit solutions for all other equilibrium objects, conditional on (α, η, κ). □

Proof of Proposition 2.5. If a steady state exists, it must satisfy $\mu^\alpha = \mu^\eta = 0$. Solve this system, conditional on κ. Denote the unique solution by $\alpha^*(\kappa)$ and $\eta^*(\kappa)$ assuming both are interior to $(0,1)$. Similarly, equation (15), which holds for any values of (α, η), gives a function $\kappa^*(\alpha, \eta)$. For $y \in [0,1]$ define $F(y) := \kappa^*(\alpha^*(y), \eta^*(y)) - y$.

Therefore, to prove a steady state exists, it suffices to prove that $F(y) = 0$ has a root in $[0,1]$ call it κ_∞. In that case, an interior steady state is given by $\alpha_\infty := \alpha^*(\kappa_\infty)$ and $\eta_\infty := \eta^*(\kappa_\infty)$.

I analyze the case $\hat{\sigma}_A > 0, \hat{\sigma}_B > 0, \phi_A \in (0,1), \phi_B \in (0,1)$. The corner cases can be analyzed in a similar fashion. The function $F(y)$ is given by

$$F(y) := -y + \min\{1,\, \max\{0,\, \tilde{\kappa}^*(y)\}\}, \tag{75}$$

where

$$\begin{aligned}
\tilde{\kappa}^*(y) &:= \frac{G_A - G_B + M_B(y)}{M_A(y) + M_B(y)} \\
M_A(y) &:= \|\sigma_A\|^2 + [\frac{(1-\phi_A)^2}{\alpha^*(y)(1-\eta^*(y))} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta^*(y)}]\hat{\sigma}_A^2 \\
M_B(y) &:= \|\sigma_B\|^2 + [\frac{(1-\phi_B)^2}{(1-\alpha^*(y))(1-\eta^*(y))} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta^*(y)}]\hat{\sigma}_B^2 \\
\alpha^*(y) &:= \frac{y(1-\phi_A)\hat{\sigma}_A}{y(1-\phi_A)\hat{\sigma}_A + (1-y)(1-\phi_B)\hat{\sigma}_B} \\
\eta^*(y) &:= \frac{\sqrt{(y\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-y)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2}}{\sqrt{(y\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-y)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2} + y(1-\phi_A)\hat{\sigma}_A + (1-y)(1-\phi_B)\hat{\sigma}_B}.
\end{aligned}$$

By inspection, $F(0) = F(1) = 0$. Thus, a steady state exists. Furthermore, if $G_A - G_B$ is sufficiently small, then it is possible to find $y^*_+ \in (0,1)$ such that $F(y^*_+) > 0$ and $y^*_- \in (0,1)$ such that $F(y^*_-) < 0$. As F is continuous, there exists interior $y^* \in (0,1)$ such that $F(y^*) = 0$. □

Lemma A.1. *Consider the steady-state equilibrium of Proposition 2.5, assuming parameters are such that* $\kappa_\infty \in$

$(0,1)$. *Let* $\mathbf{p}$ *denote any parameter. Then,*

$$\frac{d\alpha_\infty}{d\mathbf{p}} = \frac{1}{1 - D_{\alpha,\kappa}D_{\kappa,\alpha} - D_{\eta,\kappa}D_{\kappa,\eta}}\Big[(1 - D_{\eta,\kappa}D_{\kappa,\eta})C_{\alpha,\mathbf{p}} + D_{\alpha,\kappa}D_{\kappa,\eta}C_{\eta,\mathbf{p}} + D_{\alpha,\kappa}C_{\kappa,\mathbf{p}}\Big] \tag{76}$$

$$\frac{d\eta_\infty}{d\mathbf{p}} = \frac{1}{1 - D_{\alpha,\kappa}D_{\kappa,\alpha} - D_{\eta,\kappa}D_{\kappa,\eta}}\Big[D_{\eta,\kappa}D_{\kappa,\alpha}C_{\alpha,\mathbf{p}} + (1 - D_{\alpha,\kappa}D_{\kappa,\alpha})C_{\eta,\mathbf{p}} + D_{\eta,\kappa}C_{\kappa,\mathbf{p}}\Big] \tag{77}$$

$$\frac{d\kappa_\infty}{d\mathbf{p}} = \frac{1}{1 - D_{\alpha,\kappa}D_{\kappa,\alpha} - D_{\eta,\kappa}D_{\kappa,\eta}}\Big[D_{\kappa,\alpha}C_{\alpha,\mathbf{p}} + D_{\kappa,\eta}C_{\eta,\mathbf{p}} + C_{\kappa,\mathbf{p}}\Big] \tag{78}$$

where $C_{\alpha,\mathbf{p}} := \frac{\partial \alpha_\infty}{\partial \mathbf{p}}$, $C_{\eta,\mathbf{p}} := \frac{\partial \eta_\infty}{\partial \mathbf{p}}$, $C_{\kappa,\mathbf{p}} := \frac{\partial \kappa_\infty}{\partial \mathbf{p}}$,

$$\begin{aligned}
D_{\alpha,\kappa} &:= \alpha_\infty(1-\alpha_\infty)\kappa_\infty^{-1}(1-\kappa_\infty)^{-1} \\
D_{\eta,\kappa} &:= (1-\eta_\infty)\eta_\infty\hat{\pi}_\infty^{-2}\Big[\kappa_\infty^{-1}(\hat{\pi}^2_{F\to A,\infty} - \alpha_\infty\hat{\pi}^2_\infty) - (1-\kappa_\infty)^{-1}(\hat{\pi}^2_{F\to B,\infty} - (1-\alpha_\infty)\hat{\pi}^2_\infty)\Big] \\
D_{\kappa,\alpha} &:= \frac{1}{M_A + M_B}(1-\eta_\infty)\kappa_\infty^{-1}(1-\kappa_\infty)^{-1}\hat{\pi}^2_\infty \\
D_{\kappa,\eta} &:= \frac{1}{M_A + M_B}\Big[\kappa_\infty^{-1}(\hat{\pi}^2_{F\to A,\infty} - \alpha_\infty\hat{\pi}^2_\infty) - (1-\kappa_\infty)^{-1}(\hat{\pi}^2_{F\to B,\infty} - (1-\alpha_\infty)\hat{\pi}^2_\infty)\Big],
\end{aligned}$$

$\hat{\pi}_\infty = \hat{\pi}_{A,\infty} = \hat{\pi}_{B,\infty}$ *is defined by*

$$\hat{\pi}_\infty := \sqrt{(\kappa_\infty\phi_A(1-\Delta_A)\hat{\sigma}_A)^2 + ((1-\kappa_\infty)\phi_B(1-\Delta_B)\hat{\sigma}_B)^2} + \kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B$$

and where M_A, M_B *are defined by*

$$\begin{aligned}
M_A &:= \|\sigma_A\|^2 + [\frac{(1-\phi_A)^2}{\alpha_\infty(1-\eta_\infty)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta_\infty}]\hat{\sigma}_A^2 \\
M_B &:= \|\sigma_B\|^2 + [\frac{(1-\phi_B)^2}{(1-\alpha_\infty)(1-\eta_\infty)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta_\infty}]\hat{\sigma}_B^2.
\end{aligned}$$

Proof of Lemma A.1. By Proposition 2.5, there exists a steady-state. Assuming parameters are such that $\kappa_\infty \in (0,1)$, we may apply the implicit function theorem to totally differentiate the expressions relating $\alpha_\infty, \eta_\infty, \kappa_\infty$ with respect to $\mathbf{p}$. After some algebra, and using the fact that $\hat{\pi}_\infty = \hat{\pi}_{A,\infty} = \hat{\pi}_{B,\infty} = \sqrt{\hat{\pi}^2_{F\to A,\infty} + \hat{\pi}^2_{F\to B,\infty}}$, we obtain the following three-dimensional system:

$$\begin{aligned}
\frac{d\alpha_\infty}{d\mathbf{p}} &= C_{\alpha,\mathbf{p}} + D_{\alpha,\kappa}\frac{d\kappa_\infty}{d\mathbf{p}} \\
\frac{d\eta_\infty}{d\mathbf{p}} &= C_{\eta,\mathbf{p}} + D_{\eta,\kappa}\frac{d\kappa_\infty}{d\mathbf{p}} \\
\frac{d\kappa_\infty}{d\mathbf{p}} &= C_{\kappa,\mathbf{p}} + D_{\kappa,\alpha}\frac{d\alpha_\infty}{d\mathbf{p}} + D_{\kappa,\eta}\frac{d\eta_\infty}{d\mathbf{p}}
\end{aligned}$$

We may solve this system to obtain the desired result. □

Assumption 3. *Parameters are such that* $D_{\alpha,\kappa}D_{\kappa,\alpha} + D_{\eta,\kappa}D_{\kappa,\eta} < 1$ *in Lemma A.1.*

Remark 1. *Assumption 3 can be satisfied for a variety of parameter choices. Some examples are listed below (one can verify these parameter constellations satisfy Assumption 3 with simple but tedious algebra):*

(i) One sector is completely diversified, i.e., either $\Delta_A = 1$ *or* $\Delta_B = 1$ *with* any *arbitrary values for the other parameters.*

(ii) Symmetric sectors, i.e., set $G_A = G_B$, $\phi_A = \phi_B$, $\Delta_A = \Delta_B$, $\hat{\sigma}_A = \hat{\sigma}_B$, *and* $\|\sigma_A\| = \|\sigma_B\|$.

(iii) One-sector economy, i.e., $G_A = -\infty$ *or* $G_B = -\infty$ *(resulting in* $\kappa_\infty = \alpha_\infty$ *equal to 0 or 1).*

(iv) Sufficient aggregate volatility, i.e., large enough $\|\sigma_A\|$ *and* $\|\sigma_B\|$ *with* any *arbitrary values for the other parameters.*

Proof of Proposition 2.6. Using the notation of Lemma A.1, we have the partial derivatives (for any parameter values)

$$\begin{aligned} C_{\alpha,\Delta_A} &= 0 \\ C_{\eta,\Delta_A} &= -\frac{1-\eta_\infty}{\eta_\infty}(1-\Delta_A)(\kappa_\infty \phi_A \hat{\sigma}_A)^2 \hat{\pi}_\infty^{-2} \\ C_{\kappa,\Delta_A} &= \frac{2\kappa_\infty \phi_A^2 (1-\Delta_A)^2 \hat{\sigma}_A^2}{(M_A+M_B)\eta_\infty}. \end{aligned}$$

By imposing symmetric parameters, one obtains $\alpha_\infty = \kappa_\infty = 1/2$. As a result, $D_{\eta,\kappa} = D_{\kappa,\eta} = 0$ and $D_{\alpha,\kappa} = 1$ and $D_{\kappa,\alpha} = \frac{2}{M}(1-\eta_\infty)^{-1}(1-\phi)^2\hat{\sigma}^2 < 1$. Then, use expressions (77)-(78) to verify part (i) of the proposition.

Part (ii) is obtained immediately by noticing $\kappa_\infty = 1/2$ for any values of $\Delta := \Delta_A = \Delta_B$. Part (iii) is obtained immediately by noting that $\eta_\infty = 1$ under $\phi_A = \phi_B = 1$. □

Proof of Proposition 2.7. Conjecture (and verify) that $d\alpha_t = d\eta_t = 0$ in equilibrium. If so, then indirect utility is given by $\mathcal{U} = \log(n) + H$, where n is the agent's net worth and H is an occupation-specific constant. Then, agents' HJB equations, conditional on an occupation, are given by

$$\rho H = \rho \log(\rho) + r - \rho + \frac{1}{2}\|\pi\|^2 + \frac{1}{2}\|\hat{\pi}\|^2,$$

where $\hat{\pi}$ is the (shadow) vector of idiosyncratic risk prices earned by the agent. Free occupational choice implies $H_F = H_A = H_B$. Consequently, equilibrium requires $\hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B} = \hat{\pi}^2_A = \hat{\pi}^2_B$. This system of equations exactly characterizes $d\alpha_t = d\eta_t = 0$. Through the same line of arguments, it is easily verified that there cannot be any equilibrium with $d\alpha_t \neq 0$ or $d\eta_t \neq 0$. □

Proof of Proposition 3.1. In the model with distressed investors, market clearing conditions are modified from Definition 1 to the following:

- *Goods:*

$$\begin{aligned} \int_0^1 [G_A k^A_{i,t} + G_B k^B_{i,t}]di = \int_0^1 [c^A_{i,t} + c^B_{i,t} + c^F_{i,t} + c^D_{i,t}]di + \frac{1}{dt}\int_0^1 [dI^A_{i,t} + dI^B_{i,t}]di \\ + \chi \int_0^1 n^D_{i,t}(\lambda^A_{D,i,t} + \lambda^B_{D,i,t})di. \end{aligned}$$

- *Funding:*

$$\int_{i-\Delta_z}^{i} \Delta_z^{-1}[\lambda^z_{F,j,t} n^F_{i,t} + \lambda^z_{D,j,t} n^D_{i,t}]dj = \phi_z k^z_{i,t}, \quad \forall i \in [0,1], \quad z \in \{A,B\}.$$

- *Aggregate risk:*

$$\int_0^1 [\theta^A_{i,t} n^A_{i,t} + \theta^B_{i,t} n^B_{i,t} + \theta^F_{i,t} n^F_{i,t} + \theta^D_{i,t} n^D_{i,t}]di = 0.$$

- *Bond:*

$$\int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F + n_{i,t}^D]di = \int_0^1 [k_{i,t}^A + k_{i,t}^B]di.$$

In the proof below, I restrict attention to symmetric equilibria, in which all equilibrium objects are independent of i. To simplify notation, I drop all i subscripts when the meaning is clear. Within the class of symmetric equilibria, I solve for the equilibrium objects in two steps. In the first step, I assume (κ_t, ζ_t) are known and use them to solve for all other objects. In the second step, I solve for (κ_t, ζ_t) via a system of nonlinear equations. In this proof, I treat the case $\delta = 0$. The general case with $\delta > 0$ is accounted for by Lemma A.2 below.

Step 1. Solving for equilibrium given (κ, ζ).

Using the optimal consumption decisions from Appendix A.1, we can write the goods market clearing condition as (26). Thus, the aggregate investment rate ι is solved, given κ and λ_D^A, λ_D^B.

Next, apply optimal portfolio choice as special cases of the general formulas derived in Appendix A.1. If insiders can hedge their exposures to aggregate risk dZ_t, i.e., if θ_A and θ_B are unconstrained, then their optimal capital portfolio choice is given by the mean-variance portfolio

$$\frac{\kappa}{(1-\eta)\alpha} = \frac{[G_A - r - \phi_A s_A - (1-\phi_A)\sigma_A \cdot \pi]^+}{(1-\phi_A)^2 \hat{\sigma}_A^2} \quad \text{and} \quad \frac{1-\kappa}{(1-\eta)(1-\alpha)} = \frac{[G_B - r - \phi_B s_B - (1-\phi_B)\sigma_B \cdot \pi]^+}{(1-\phi_B)^2 \hat{\sigma}_B^2}. \tag{79}$$

If insiders cannot hedge, i.e., if $\theta_A \equiv \theta_B \equiv 0$ are constrained, then their optimal capital portfolio choice is given by

$$\frac{\kappa}{(1-\eta)\alpha} = \frac{[G_A - r - \phi_A s_A]^+}{(1-\phi_A)^2(\hat{\sigma}_A^2 + \|\sigma_A\|^2)} \quad \text{and} \quad \frac{1-\kappa}{(1-\eta)(1-\alpha)} = \frac{[G_B - r - \phi_B s_B]^+}{(1-\phi_B)^2(\hat{\sigma}_B^2 + \|\sigma_B\|^2)}. \tag{80}$$

In the hedging case, define $\pi_A := \pi$ and $\pi_B := \pi$. In the non-hedging case, define $\pi_A := \hat{\pi}_A \sigma_A / \hat{\sigma}_A$ and $\pi_B := \hat{\pi}_B \sigma_B / \hat{\sigma}_B$ where $\hat{\pi}_A = \kappa(1-\phi_A)\hat{\sigma}_A/(1-\eta)\alpha$ and $\hat{\pi}_B = (1-\kappa)(1-\phi_B)\hat{\sigma}_B/(1-\eta)(1-\alpha)$ are defined by (13). With this notation, (79) and (80) are both equivalent to the following more general expressions, which also account for the shorting constraints:

$$(1-\phi_A)[\hat{\sigma}_A \hat{\pi}_A + \sigma_A \cdot \pi_A] \geq G_A - r - \phi_A s_A \quad \text{with equality when } \kappa > 0 \tag{81}$$

$$(1-\phi_B)[\hat{\sigma}_B \hat{\pi}_B + \sigma_B \cdot \pi_B] \geq G_B - r - \phi_B s_B \quad \text{with equality when } \kappa < 1. \tag{82}$$

By taking the difference between (81) and (82), and noting that at least one of these is always an equality, we obtain

$$0 = \min(1-\kappa, \max(-\kappa, H)), \tag{83}$$

where H is defined in (27). Equation (83) can be re-written as (27). The interest rate r can be determined by multiplying (81) and (82) by κ and $1-\kappa$, respectively, and summing the conditions.

Exposures to aggregate risk dZ_t are determined as follows. Due to log utility, π_A, π_B represent insiders' optimal exposures regardless of whether or not insiders can trade aggregate risk. Thus, insiders set $\theta_A + (1-\phi_A)(1-\eta)^{-1}\alpha^{-1}\kappa\sigma_A = \pi_A$ and $\theta_B + (1-\phi_B)(1-\eta)^{-1}(1-\alpha)^{-1}(1-\kappa)\sigma_B = \pi_B$. For financiers and distressed investors, who can always trade aggregate risk without constraints, their optimal exposure is equal to the aggregate risk price vector π. Thus, $\theta_F + \lambda_F^A \sigma_A + \lambda_F^B \sigma_B = \theta_D + \lambda_D^A \sigma_A + \lambda_D^B \sigma_B = \pi$.

In equilibrium, the aggregate risk price vector is determined by applying aggregate risk market clearing,

which can be restated as

$$\eta[x\pi + (1-x)\pi] + (1-\eta)[\alpha\pi_A + (1-\alpha)\pi_B] = \kappa\sigma_A + (1-\kappa)\sigma_B.$$

If insiders can trade aggregate risk without constraints ($\pi_A = \pi_B = \pi$), then we solve for $\pi = \kappa\sigma_A + (1-\kappa)\sigma_B$. If insiders cannot trade aggregate risk ($\pi_A = \hat{\pi}_A\sigma_A/\hat{\sigma}_A$ and $\pi_B = \hat{\pi}_B\sigma_B/\hat{\sigma}_B$), then we use the definitions of $\hat{\pi}_A, \hat{\pi}_B$ in (13) to solve for $\pi = \eta^{-1}[\kappa\phi_A\sigma_A + (1-\kappa)\phi_B\sigma_B]$.

Next, we determine spreads by using funding market clearing, which aggregates to

$$x\lambda_F^A + (1-x)\lambda_D^A = \frac{\kappa\phi_A}{\eta} \quad \text{and} \quad x\lambda_F^B + (1-x)\lambda_D^B = \frac{(1-\kappa)\phi_B}{\eta}.$$

Substituting the optimal positions (23) and (25), we have

$$x(s_A - \sigma_A\cdot\pi - \zeta)^+ + (1-x)(s_A - \sigma_A\cdot\pi - \chi)^+ = \frac{\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2}{\eta} := x(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A},$$

and symmetrically for sector B. Note that $s_A - \sigma_A\cdot\pi > \min(\zeta,\chi)$ is required for this equation to hold. The remaining exhaustive cases are as follows. If $s_A - \sigma_A\cdot\pi \geq \max(\zeta,\chi)$, then

$$s_A - \sigma_A\cdot\pi = x\zeta + (1-x)\chi + x(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\zeta - \chi - \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \leq 0$ and $\chi - \zeta - (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \leq 0$, which implies (28) holds. If $\chi \geq s_A - \sigma_A\cdot\pi > \zeta$, then

$$s_A - \sigma_A\cdot\pi = \zeta + (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\chi - \zeta - (1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \geq 0$, which implies (28) holds. If $\zeta \geq s_A - \sigma_A\cdot\pi > \chi$, then

$$s_A - \sigma_A\cdot\pi = \chi + \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}.$$

Thus, this case obtains when $\zeta - \chi - \frac{x}{1-x}(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \geq 0$, which implies (28) holds. An identical analysis holds for sector B. Combining these results, equation (28) holds in all cases.

Finally, we determine state variable dynamics. Define the following expressions:

$$\begin{aligned}
\Pi_F &:= \theta_F\cdot\pi + \lambda_F^A s_A + \lambda_F^B s_B \\
\Pi_D &:= \theta_D\cdot\pi + \lambda_D^A(s_A - \chi) + \lambda_D^B(s_B - \chi) \\
\Pi_A &:= \theta_A\cdot\pi + (1-\eta)^{-1}\alpha^{-1}\kappa(G_A - r - \phi_A s_A) \\
\Pi_B &:= \theta_B\cdot\pi + (1-\eta)^{-1}(1-\alpha)^{-1}(1-\kappa)(G_B - r - \phi_B s_B).
\end{aligned}$$

Note that $K_A/N_A = \kappa/(1-\eta)\alpha$ and $K_B/N_B = (1-\kappa)/(1-\eta)(1-\alpha)$ by bond market clearing. Thus, aggregating agents' net worth evolutions, which eliminates any contributions from $\{dW_{i,t}\}_{i\in[0,1]}$ due to Lemma

2.2, and substituting these expressions for $\Pi_F, \Pi_D, \Pi_A, \Pi_B$, we obtain

$$\frac{dN_F}{N_F} = \big[r - \rho_F + \Pi_F\big]dt + \big[\theta_F + \lambda_F^A\sigma_A + \lambda_F^B\sigma_B\big]dZ_t \tag{84}$$

$$\frac{dN_D}{N_D} = \big[r - \rho + \Pi_D\big]dt + \big[\theta_D + \lambda_D^A\sigma_A + \lambda_D^B\sigma_B\big]dZ_t \tag{85}$$

$$\frac{dN_A}{N_A} = \big[r - \rho + \Pi_A\big]dt + \big[\theta_A + (1-\eta)^{-1}\alpha^{-1}\kappa\sigma_A\big]dZ_t \tag{86}$$

$$\frac{dN_B}{N_B} = \big[r - \rho + \Pi_B\big]dt + \big[\theta_B + (1-\eta)^{-1}(1-\alpha)^{-1}(1-\kappa)\sigma_B\big]dZ_t. \tag{87}$$

Using expressions (81)-(82), we have shown that $\Pi_F, \Pi_D, \Pi_A, \Pi_B$ are equivalent to the expressions in (29), (30), and (31) of Proposition 3.1. Furthermore, using the net worth evolutions in (84)-(86) and substituting the optimal aggregate risk exposures derived above, then applying Itô's formula to the definitions of (α, η, x), we obtain the state variable evolutions in (32)-(37) of Proposition 3.1.

Step 2. Solving for (κ, ζ).

Substituting π, (23), (28) into (24), we obtain a single equation in (κ, ζ). Substituting π, π_A, π_B, (28) into (27), we obtain a second equation in (κ, ζ). A solution exists, as demonstrated by Proposition A.3. □

Proof of Proposition 3.2. First, note that the stated assumptions imply $\kappa = \alpha = 1/2$ at all times. Next, $\Delta = \Delta_\tau$ large enough implies the leverage constraint (22) binds in the future. Indeed, Δ large enough guarantees

$$\chi > \frac{1}{2}\bar{\lambda}(1-\Delta)^2\hat{\sigma}^2 \tag{88}$$

holds. Using the thresholds computed in Proposition 3.4, as well as part (i) of Proposition 3.5, inequality (88) implies that $\lambda_D^A + \lambda_D^B > 0$ if and only if $\lambda_F^A + \lambda_F^B = \bar{\lambda}$. Hence, computing the equilibrium drift of $\log(x\eta)$ away from the constraint, using the fact that the drift computed using $\lambda_D^A = \lambda_D^B = 0$, we have

$$\begin{aligned}
\mu^{\log(x\eta)} &= (1-\eta)[x(\rho-\rho_F) + \frac{x}{2}(\frac{\phi(1-\Delta)\hat{\sigma}}{x\eta})^2 - (\frac{(1-\phi)\hat{\sigma}}{1-\eta})^2] + \frac{\delta}{\eta}(\nu_F + \nu_D - \eta) \\
&\quad + (1-x)[\rho - \rho_F + \frac{1}{2}(\frac{\phi(1-\Delta)\hat{\sigma}}{x\eta})^2] + \frac{\delta}{x}((\nu_F+\nu_D)^{-1}\nu_F - x) \\
&= (1-x\eta)(\rho-\rho_F) + \frac{1}{2}(1-x\eta)(\frac{\phi(1-\Delta)\hat{\sigma}}{x\eta})^2 - \frac{(1-\phi)^2\hat{\sigma}^2}{1-\eta} + \frac{\delta}{x\eta}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D) - 2x\eta].
\end{aligned}$$

Note that

$$\sup_{x,\eta\,:\,x\eta<\phi/\bar{\lambda}}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D)] = \max_{x,\eta\,:\,x\eta=\phi/\bar{\lambda}}[\eta\frac{\nu_F}{\nu_F+\nu_D} + x(\nu_F+\nu_D)] = \frac{2(\nu_F\phi/\bar{\lambda})^{1/2}}{\nu_F+\nu_D}.$$

Also using the facts that $x \leq 1$, $x\eta > \phi/\bar{\lambda}$, and $\chi > \frac{1}{2}\frac{\phi(1-\Delta)^2\hat{\sigma}^2}{x\eta}$ away from the constraint, we have

$$\begin{aligned}
\mu^{\log(x\eta)} &< (1-x\eta)(\bar{\lambda}\chi + \rho - \rho_F) - \frac{(1-\phi)^2\hat{\sigma}^2}{1-x\eta} + \frac{2\delta}{\nu_F+\nu_D}[(\nu_F\bar{\lambda}/\phi)^{1/2} - \nu_F - \nu_D] \\
&< \frac{1}{1-x\eta}\Big[(1-x\eta)^2(\bar{\lambda}\chi + \rho - \rho_F) - (1-\phi)^2\hat{\sigma}^2 + \frac{2\delta(1-x\eta)}{\nu_F+\nu_D}[(\nu_F\bar{\lambda}/\phi)^{1/2} - \nu_F - \nu_D]\Big]
\end{aligned}$$

Maximizing over $x\eta \geq \phi/\bar{\lambda}$ and using $(1-\phi/\bar{\lambda})^2(\bar{\lambda}\chi + \rho - \rho_F) + (1-\phi)^2\hat{\sigma}^2 < 0$, the corner solution $x\eta = \phi/\bar{\lambda}$

must hold, so

$$\mu^{\log(x\eta)} < \frac{1}{1-\phi/\bar{\lambda}}\Big[(1-\phi/\bar{\lambda})^2(\bar{\lambda}\chi+\rho-\rho_F)-(1-\phi)^2\hat{\sigma}^2+\frac{2\delta(1-\phi/\bar{\lambda})}{\nu_F+\nu_D}[(\nu_F\bar{\lambda}/\phi)^{1/2}-\nu_F-\nu_D]\Big]<0$$

by (39). This shows that if Δ_τ is made large enough, then $\log(x_{\tau+t}\eta_{\tau+t})$ must have a negative drift at least until the leverage constraint (22) is hit, i.e., $x_{\tau+T}\eta_{\tau+T}=\phi/\bar{\lambda}$. Suppose the constraint is hit at time $\tau+T$.

Using Proposition A.3, we may compute

$$\zeta_{\tau+T}=\chi-\frac{1}{2}\bar{\lambda}(1-\Delta)^2\hat{\sigma}^2>0$$

by (88). We may then repeat calculations similar to those above, substituting $x\eta=\phi/\bar{\lambda}$, to find that $\mu_{\tau+T}^{\log(x\eta)}<0$ under condition (39) as well. Equity market clearing says $\eta(1-x)\lambda_D^A=\eta(1-x)\lambda_D^B=\frac{1}{2}\phi-\eta x\frac{1}{2}\bar{\lambda}$. Using (28), we have that spreads are equal to χ so that $\lambda_{D,\tau+T}^A=\lambda_{D,\tau+T}^B=0$. Thus, $d\lambda_{D,\tau+T}^A>0$ and $d\lambda_{D,\tau+T}^B>0$, which implies distress costs

$$\chi\eta_{\tau+T+\varepsilon}(1-x_{\tau+T+\varepsilon})(\lambda_{D,\tau+T+\varepsilon}^A+\lambda_{D,\tau+T+\varepsilon}^B)>0$$

for some ε small enough. This proves that (38) holds. □

Proof of Proposition 3.3. First, note that $Y=G-\chi\eta(1-x)(\lambda_D^A+\lambda_D^B)$ so that

$$\text{Var}_t[dY_t]=\chi^2\Big(\eta_t^2(1-x_t)^2\text{Var}_t[d(\lambda_{D,t}^A+\lambda_{D,t}^B)]+(\lambda_{D,t}^A+\lambda_{D,t}^B)^2\text{Var}_t[d(\eta_t(1-x_t))]\Big).$$

With Δ_A,Δ_B chosen high enough, the conditions of Proposition 3.5 hold so that $\lambda_D^A+\lambda_D^B>0$ if and only if $\lambda_F^A+\lambda_F^B=\bar{\lambda}$. Therefore, $\text{Var}_t[dY_t]=0$ when the leverage constraint is slack and $\text{Var}_t[dY_t]>0$ otherwise. □

Proof of Proposition 3.4. Let $\bar{\lambda}=+\infty$ so that $\zeta=0$ in Proposition 3.1. Specializing (28) to this case, we have Then, substituting s_A and π into λ_D^A in (25), we find $\lambda_D^A>0$ if and only if $(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A}>\chi$. Substituting $\hat{\pi}_{F\to A}:=\kappa\phi_A(1-\Delta_A)\hat{\sigma}_A/(x\eta)$, this implies $\omega_A^*:=\chi^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2>x\eta$. An identical analysis holds for sector B. □

Proof of Proposition 3.5. Suppose $\lambda_F^A+\lambda_F^B=\bar{\lambda}$, but $\lambda_D^A+\lambda_D^B=0$. The latter, plus funding market clearing, implies $\lambda_F^A=\kappa\phi_A/x\eta$ and $\lambda_F^B=(1-\kappa)\phi_B/x\eta$. Summing these results yields $\kappa\phi_A+(1-\kappa)\phi_B=\bar{\lambda}x\eta$. If $\phi_A=\phi_B$, then this implies $x\eta=\bar{\lambda}^{-1}$, which contradicts the diffusive nature of $x_t\eta_t$. This proves part (i).

Now, suppose $\lambda_F^A+\lambda_F^B<\bar{\lambda}$ so that $\zeta=0$. Then, using the stated assumption of part (ii) and equation (23), we have $\chi\geq\bar{\lambda}(1-\Delta_A)^2\hat{\sigma}_A^2\geq\lambda_F^A(1-\Delta_A)^2\hat{\sigma}_A^2\geq s_A-\sigma_A\cdot\pi$, so that $\lambda_D^A=0$ by (25). Repeating this analysis for sector B proves part (ii). □

Lemma A.2. *Consider the OLG framework of Section 3. Equilibrium holds with subjective discount rate ρ_z for agent $z\in\{A,B,F,D\}$ replaced by $\tilde{\rho}_z:=\rho_z+\delta$, and with μ^α, μ^η, and μ^x augmented with $\delta((\nu_A+\nu_B)^{-1}\nu_A-\alpha)$, $\delta(\nu_F+\nu_D-\eta)$, and $\delta((\nu_F+\nu_D)^{-1}\nu_F-x)$ i.e., replaced by*

$$\begin{aligned}\mu^\alpha&=\mu_0^\alpha+\delta((\nu_A+\nu_B)^{-1}\nu_A-\alpha)\\ \mu^\eta&=\mu_0^\eta+\delta(\nu_F+\nu_D-\eta)\\ \mu^x&=\mu_0^x+\delta((\nu_F+\nu_D)^{-1}\nu_F-x),\end{aligned}$$

where $\mu_0^\alpha,\mu_0^\eta,\mu_0^x$ come from the economy with $\delta=0$.

Proof of Lemma A.2. A proof of the fact that subjective discount rates in recursive preferences are simply augmented by the Poisson death rate can be found in the appendix of Gârleanu and Panageas (2015). OLG adds the following terms to the dynamics of aggregate net worth:

$$dN_{z,t} = \ldots - \delta N_{z,t} dt + \nu_z \delta K_t dt.$$

Applying Itô's formula to the wealth shares (α, η, x) yields the result on the state drifts. □

Proposition A.3. *In the equilibrium of Proposition 3.1, the solution (κ, ζ) to (24) and (27) is determined as follows. Consider the state space $\Omega := [0,1]^3$ for (α, η, x). First define the following objects on Ω:*

$$\Gamma_A = \begin{cases} 1, & \text{if insiders may frictionlessly trade aggregate risk (unconstrained } \theta_A, \theta_B) \\ \frac{\phi_A^2}{\eta} + \frac{(1-\phi_A)^2}{\alpha(1-\eta)}, & \text{if insiders may not trade aggregate risk } (\theta_A = \theta_B \equiv 0), \end{cases}$$

$$\Gamma_B = \begin{cases} 1, & \text{if insiders may frictionlessly trade aggregate risk (unconstrained } \theta_A, \theta_B) \\ \frac{\phi_B^2}{\eta} + \frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)}, & \text{if insiders may not trade aggregate risk } (\theta_A = \theta_B \equiv 0), \end{cases}$$

and

$$\begin{aligned} \Sigma_A &:= \frac{(1-\Delta_A)^2 \hat{\sigma}_A^2}{\eta(1-x)} \\ \Sigma_B &:= \frac{(1-\Delta_B)^2 \hat{\sigma}_B^2}{\eta(1-x)} \\ M_A &:= \Gamma_A \|\sigma_A\|^2 + \big[\frac{(1-\phi_A)^2}{\alpha(1-\eta)} + \frac{\phi_A^2(1-\Delta_A)^2}{\eta}\big]\hat{\sigma}_A^2 \\ M_B &:= \Gamma_B \|\sigma_B\|^2 + \big[\frac{(1-\phi_B)^2}{(1-\alpha)(1-\eta)} + \frac{\phi_B^2(1-\Delta_B)^2}{\eta}\big]\hat{\sigma}_B^2. \end{aligned}$$

Next, for each $(\alpha, \eta, x) \in \Omega$, define the following function mapping $[0,1] \mapsto \mathbb{R}$:

$$\begin{aligned} \tilde{\lambda}(\kappa) := \mathbf{1}_{\{\Delta_A = 1\}} \frac{\kappa \phi_A}{x\eta} + \mathbf{1}_{\{\Delta_A < 1\}} \Big\{ x \frac{\kappa \phi_A}{x\eta} + (1-x) \frac{\chi}{(1-\Delta_A)^2 \hat{\sigma}_A^2} - (1-x) \Big[\frac{\chi}{(1-\Delta_A)^2 \hat{\sigma}_A^2} - \frac{\kappa \phi_A}{x\eta} \Big]^+ \Big\} \\ + \mathbf{1}_{\{\Delta_B = 1\}} \frac{(1-\kappa)\phi_B}{x\eta} + \mathbf{1}_{\{\Delta_B < 1\}} \Big\{ x \frac{(1-\kappa)\phi_B}{x\eta} + (1-x) \frac{\chi}{(1-\Delta_B)^2 \hat{\sigma}_B^2} - (1-x) \Big[\frac{\chi}{(1-\Delta_B)^2 \hat{\sigma}_B^2} - \frac{(1-\kappa)\phi_B}{x\eta} \Big]^+ \Big\}. \end{aligned}$$

Also define the following functions mapping $\mathbb{R} \mapsto \mathbb{R}$*:*

$$\begin{aligned}
\tilde{\kappa}(\zeta) &:= \frac{G_A - G_B + M_B - (\phi_A - \phi_B)(x\zeta + (1-x)\chi)}{M_A + M_B} \\
\tilde{\kappa}_{F\not\to A}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A\chi + \phi_B(x\zeta + (1-x)\chi)}{M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2} \\
\tilde{\kappa}_{F\not\to B}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A(x\zeta + (1-x)\chi) + \phi_B\chi + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2} \\
\tilde{\kappa}_{D\not\to A}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A\zeta + \phi_B(x\zeta + (1-x)\chi)}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2} \\
\tilde{\kappa}_{D\not\to B}(\zeta) &:= \frac{G_A - G_B + M_B - \phi_A(x\zeta + (1-x)\chi) + \phi_B\zeta + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2} \\
\tilde{\kappa}_{D\not\to A,B}(\zeta) &:= \frac{G_A - G_B + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}{M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2 + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2}[1 - \mathbf{1}_{\{\zeta>0,\phi_A\neq\phi_B\}}] \\
&\quad + \frac{x\eta\bar{\lambda} - \phi_B}{\phi_A - \phi_B}\mathbf{1}_{\{\zeta>0,\phi_A\neq\phi_B\}}.
\end{aligned}$$

Using these functions, for each $(\alpha, \eta, x) \in \Omega$*, define the following objects:*

$$\begin{aligned}
\tilde{\kappa}_0^* &:= [\tilde{\kappa}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to A}^* &:= [\tilde{\kappa}_{D\not\to A}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to B}^* &:= [\tilde{\kappa}_{D\not\to B}(0)]^+ \wedge 1 \\
\tilde{\kappa}_{0,D\not\to A,B}^* &:= [\tilde{\kappa}_{D\not\to A,B}(0)]^+ \wedge 1,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\zeta}^* &:= \frac{(\Sigma_A^{-1} + \Sigma_B^{-1})\chi + \tilde{\kappa}(0)\phi_A + (1 - \tilde{\kappa}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_A^{-1} + \Sigma_B^{-1} + x[M_A + M_B]^{-1}(\phi_A - \phi_B)^2} \\
\tilde{\zeta}_{F\not\to A}^* &:= \frac{\Sigma_B^{-1}\chi + (1 - \tilde{\kappa}_{F\not\to A}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_B^{-1} + x[M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2]^{-1}\phi_B^2} \\
\tilde{\zeta}_{F\not\to B}^* &:= \frac{\Sigma_A^{-1}\chi + \tilde{\kappa}_{F\not\to B}(0)\phi_A - \eta\bar{\lambda}}{\Sigma_A^{-1} + x[M_A + M_B + x(1-x)^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2]^{-1}\phi_A^2} \\
\tilde{\zeta}_{D\not\to A}^* &:= \frac{\Sigma_B^{-1}\chi + \tilde{\kappa}_{D\not\to A}(0)x^{-1}\phi_A + (1 - \tilde{\kappa}_{D\not\to A}(0))\phi_B - \eta\bar{\lambda}}{\Sigma_B^{-1} + x[M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_A^2(1-\Delta_A)^2\hat{\sigma}_A^2]^{-1}(x^{-1}\phi_A - \phi_B)^2} \\
\tilde{\zeta}_{D\not\to B}^* &:= \frac{\Sigma_A^{-1}\chi + \tilde{\kappa}_{D\not\to B}(0)\phi_A + (1 - \tilde{\kappa}_{D\not\to B}(0))x^{-1}\phi_B - \eta\bar{\lambda}}{\Sigma_A^{-1} + x[M_A + M_B + (1-x)x^{-1}\eta^{-1}\phi_B^2(1-\Delta_B)^2\hat{\sigma}_B^2]^{-1}(\phi_A - x^{-1}\phi_B)^2} \\
\tilde{\zeta}_{D\not\to A,B}^* &:= \frac{\mathbf{1}_{\{\phi_A\neq\phi_B\}}}{(\phi_A - \phi_B)^2}\Big[(G_A - G_B)(\phi_A - \phi_B) + \Big(M_B + \frac{1-x}{x}\frac{\phi_B^2(1-\Delta_B)^2}{\eta}\hat{\sigma}_B^2\Big)(\phi_A - x\eta\bar{\lambda}) \\
&\quad + \Big(M_A + \frac{1-x}{x}\frac{\phi_A^2(1-\Delta_A)^2}{\eta}\hat{\sigma}_A^2\Big)(\phi_B - x\eta\bar{\lambda})\Big],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\kappa}^*_{\zeta} &:= \tilde{\kappa}(\tilde{\zeta}^*) \\
\tilde{\kappa}^*_{\zeta,F\not\to A} &:= \tilde{\kappa}_{F\not\to A}(\tilde{\zeta}^*_{F\not\to A}) \\
\tilde{\kappa}^*_{\zeta,F\not\to B} &:= \tilde{\kappa}_{F\not\to B}(\tilde{\zeta}^*_{F\not\to B}) \\
\tilde{\kappa}^*_{\zeta,D\not\to A} &:= \tilde{\kappa}_{D\not\to A}(\tilde{\zeta}^*_{D\not\to A}) \\
\tilde{\kappa}^*_{\zeta,D\not\to B} &:= \tilde{\kappa}_{D\not\to B}(\tilde{\zeta}^*_{D\not\to B}) \\
\tilde{\kappa}^*_{\zeta,D\not\to A,B} &:= \tilde{\kappa}_{D\not\to A,B}(\tilde{\zeta}^*_{D\not\to A,B}),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\zeta}^{\kappa=1} &:= \chi + (\Sigma_A^{-1} + \Sigma_B^{-1})^{-1}[\phi_A - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=1}_{D\not\to A} &:= \chi + \Sigma_B[x^{-1}\phi_A - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=0} &:= \chi + (\Sigma_A^{-1} + \Sigma_B^{-1})^{-1}[\phi_B - \eta\bar{\lambda}] \\
\tilde{\zeta}^{\kappa=0}_{D\not\to B} &:= \chi + \Sigma_A[x^{-1}\phi_B - \eta\bar{\lambda}].
\end{aligned}$$

Finally, for each $(\alpha,\eta,x)\in\Omega$*, define the following functions mapping* $[0,1]\times\mathbb{R}\mapsto\mathbb{R}$*:*

$$\begin{aligned}
\tilde{d}_A(\kappa,\zeta) &:= (x\eta)^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 + \zeta - \chi \\
\tilde{d}_B(\kappa,\zeta) &:= (x\eta)^{-1}(1-\kappa)\phi_B(1-\Delta_B)^2\hat{\sigma}_B^2 + \zeta - \chi \\
\tilde{f}_A(\kappa,\zeta) &:= ((1-x)\eta)^{-1}\kappa\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2 - \zeta + \chi \\
\tilde{f}_B(\kappa,\zeta) &:= ((1-x)\eta)^{-1}(1-\kappa)\phi_B(1-\Delta_B)^2\hat{\sigma}_B^2 - \zeta + \chi.
\end{aligned}$$

Using all the definitions above, construct the following subsets of Ω*:*

$$\begin{aligned}
\Omega_0 &:= \left\{(\alpha,\eta,x)\in\Omega : \tilde{d}_A(\tilde{\kappa}^*_0,0)\wedge\tilde{d}_B(\tilde{\kappa}^*_0,0)\geq 0,\ \tilde{\lambda}(\tilde{\kappa}^*_0)\leq\bar{\lambda}\right\} \\
\Omega_{0,D\not\to A} &:= \left\{(\alpha,\eta,x)\in\Omega : \tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to A},0)<0\leq\tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to A},0),\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to A})\leq\bar{\lambda}\right\} \\
\Omega_{0,D\not\to B} &:= \left\{(\alpha,\eta,x)\in\Omega : \tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to B},0)<0\leq\tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to B},0),\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to B})\leq\bar{\lambda}\right\} \\
\Omega_{0,D\not\to A,B} &:= \left\{(\alpha,\eta,x)\in\Omega : \tilde{d}_A(\tilde{\kappa}^*_{0,D\not\to A,B},0)\vee\tilde{d}_B(\tilde{\kappa}^*_{0,D\not\to A,B},0)<0,\ \tilde{\lambda}(\tilde{\kappa}^*_{0,D\not\to A,B})\leq\bar{\lambda}\right\} \\
\Omega_1 &:= \Omega\backslash(\Omega_0\cup\Omega_{0,D\not\to A}\cup\Omega_{0,D\not\to B}\cup\Omega_{0,D\not\to A,B}),
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{\zeta} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_A(\tilde{\kappa}^*_{\zeta},\tilde{\zeta}^*)\wedge\tilde{d}_B(\tilde{\kappa}^*_{\zeta},\tilde{\zeta}^*)\geq 0,\\
&\qquad \tilde{f}_A(\tilde{\kappa}^*_{\zeta},\tilde{\zeta}^*)\wedge\tilde{f}_B(\tilde{\kappa}^*_{\zeta},\tilde{\zeta}^*)\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta}\leq 1,\,\tilde{\zeta}^*>0\big\}\\
\Omega_{\zeta,D\not\to A} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to A},\tilde{\zeta}^*_{D\not\to A})<0\leq\tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to A},\tilde{\zeta}^*_{D\not\to A}),\\
&\qquad \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to A},\tilde{\zeta}^*_{D\not\to A})\wedge\tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to A},\tilde{\zeta}^*_{D\not\to A})\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta,D\not\to A}\leq 1,\,\tilde{\zeta}^*_{D\not\to A}>0\big\}\\
\Omega_{\zeta,D\not\to B} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to B},\tilde{\zeta}^*_{D\not\to B})<0\leq\tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to B},\tilde{\zeta}^*_{D\not\to B}),\\
&\qquad \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to B},\tilde{\zeta}^*_{D\not\to B})\wedge\tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to B},\tilde{\zeta}^*_{D\not\to B})\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta,D\not\to B}\leq 1,\,\tilde{\zeta}^*_{D\not\to B}>0\big\}\\
\Omega_{\zeta,D\not\to A,B} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_A(\tilde{\kappa}^*_{\zeta,D\not\to A,B},\tilde{\zeta}^*_{D\not\to A,B})\vee\tilde{d}_B(\tilde{\kappa}^*_{\zeta,D\not\to A,B},\tilde{\zeta}^*_{D\not\to A,B})<0,\\
&\qquad \tilde{f}_A(\tilde{\kappa}^*_{\zeta,D\not\to A,B},\tilde{\zeta}^*_{D\not\to A,B})\wedge\tilde{f}_B(\tilde{\kappa}^*_{\zeta,D\not\to A,B},\tilde{\zeta}^*_{D\not\to A,B})\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta,D\not\to A,B}\leq 1,\,\tilde{\zeta}^*_{D\not\to A,B}>0\big\}\\
\Omega_{\zeta,F\not\to A} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{f}_A(\tilde{\kappa}^*_{\zeta,F\not\to A},\tilde{\zeta}^*_{F\not\to A})<0\leq\tilde{f}_B(\tilde{\kappa}^*_{\zeta,F\not\to A},\tilde{\zeta}^*_{F\not\to A}),\\
&\qquad \tilde{d}_A(\tilde{\kappa}^*_{\zeta,F\not\to A},\tilde{\zeta}^*_{F\not\to A})\wedge\tilde{d}_B(\tilde{\kappa}^*_{\zeta,F\not\to A},\tilde{\zeta}^*_{F\not\to A})\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta,F\not\to A}\leq 1,\,\tilde{\zeta}^*_{F\not\to A}>0\big\}\\
\Omega_{\zeta,F\not\to B} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{f}_B(\tilde{\kappa}^*_{\zeta,F\not\to B},\tilde{\zeta}^*_{F\not\to B})<0\leq\tilde{f}_A(\tilde{\kappa}^*_{\zeta,F\not\to B},\tilde{\zeta}^*_{F\not\to B}),\\
&\qquad \tilde{d}_A(\tilde{\kappa}^*_{\zeta,F\not\to B},\tilde{\zeta}^*_{F\not\to B})\wedge\tilde{d}_B(\tilde{\kappa}^*_{\zeta,F\not\to B},\tilde{\zeta}^*_{F\not\to B})\geq 0,\, 0\leq\tilde{\kappa}^*_{\zeta,F\not\to B}\leq 1,\,\tilde{\zeta}^*_{F\not\to B}>0\big\}
\end{aligned}$$

and

$$\begin{aligned}
\Omega^{\kappa=1}_{\zeta} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_A(1,\tilde{\zeta}^{\kappa=1})\geq 0,\,\tilde{f}_A(1,\tilde{\zeta}^{\kappa=1})\geq 0,\,\tilde{\kappa}(\tilde{\zeta}^{\kappa=1})>1,\,\tilde{\zeta}^{\kappa=1}>0\big\}\\
\Omega^{\kappa=1}_{\zeta,D\not\to A} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_A(1,\tilde{\zeta}^{\kappa=1}_{D\not\to A})<0,\,\tilde{f}_A(1,\tilde{\zeta}^{\kappa=1}_{D\not\to A})\geq 0,\,\tilde{\kappa}_{D\not\to A}(\tilde{\zeta}^{\kappa=1}_{D\not\to A})>1,\,\tilde{\zeta}^{\kappa=1}_{D\not\to A}>0\big\}\\
\Omega^{\kappa=0}_{\zeta} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_B(0,\tilde{\zeta}^{\kappa=0})\geq 0,\,\tilde{f}_B(0,\tilde{\zeta}^{\kappa=0})\geq 0,\,\tilde{\kappa}(\tilde{\zeta}^{\kappa=0})<0,\,\tilde{\zeta}^{\kappa=0}>0\big\}\\
\Omega^{\kappa=0}_{\zeta,D\not\to B} &:= \big\{(\alpha,\eta,x)\in\Omega_1 : \tilde{d}_B(0,\tilde{\zeta}^{\kappa=0}_{D\not\to B})<0,\,\tilde{f}_B(0,\tilde{\zeta}^{\kappa=0}_{D\not\to B})\geq 0,\,\tilde{\kappa}_{D\not\to B}(\tilde{\zeta}^{\kappa=0}_{D\not\to B})<0,\,\tilde{\zeta}^{\kappa=0}_{D\not\to B}>0\big\}.
\end{aligned}$$

Then, the solutions for $\kappa:\Omega\mapsto\mathcal{O}$ and $\zeta:\Omega\mapsto\mathcal{O}$, where $\mathcal{O}$ are the finite subsets of $\mathbb{R}_+$, are[43]

$$\kappa\ni\begin{cases}
\tilde{\kappa}^*_0, & \text{on}\quad \Omega_0\\
\tilde{\kappa}^*_{0,D\not\to A}, & \text{on}\quad \Omega_{0,D\not\to A}\\
\tilde{\kappa}^*_{0,D\not\to B}, & \text{on}\quad \Omega_{0,D\not\to B}\\
\tilde{\kappa}^*_{0,D\not\to A,B}, & \text{on}\quad \Omega_{0,D\not\to A,B}\\
\tilde{\kappa}^*_{\zeta}, & \text{on}\quad \Omega_{\zeta}\\
\tilde{\kappa}^*_{\zeta,D\not\to A}, & \text{on}\quad \Omega_{\zeta,D\not\to A}\\
\tilde{\kappa}^*_{\zeta,D\not\to B}, & \text{on}\quad \Omega_{\zeta,D\not\to B}\\
\tilde{\kappa}^*_{\zeta,D\not\to A,B}, & \text{on}\quad \Omega_{\zeta,D\not\to A,B}\\
\tilde{\kappa}^*_{\zeta,F\not\to A}, & \text{on}\quad \Omega_{\zeta,F\not\to A}\\
\tilde{\kappa}^*_{\zeta,F\not\to B}, & \text{on}\quad \Omega_{\zeta,F\not\to B}\\
1, & \text{on}\quad \Omega^{\kappa=1}_{\zeta}\cup\Omega^{\kappa=1}_{\zeta,D\not\to A}\\
0, & \text{on}\quad \Omega^{\kappa=0}_{\zeta}\cup\Omega^{\kappa=0}_{\zeta,D\not\to B}
\end{cases}
\qquad\text{and}\qquad
\zeta\ni\begin{cases}
\tilde{\zeta}^*, & \text{on}\quad \Omega_{\zeta}\\
\tilde{\zeta}^*_{D\not\to A}, & \text{on}\quad \Omega_{\zeta,D\not\to A}\\
\tilde{\zeta}^*_{D\not\to B}, & \text{on}\quad \Omega_{\zeta,D\not\to B}\\
\tilde{\zeta}^*_{D\not\to A,B}, & \text{on}\quad \Omega_{\zeta,D\not\to A,B}\\
\tilde{\zeta}^*_{F\not\to A}, & \text{on}\quad \Omega_{\zeta,F\not\to A}\\
\tilde{\zeta}^*_{F\not\to B}, & \text{on}\quad \Omega_{\zeta,F\not\to B}\\
\tilde{\zeta}^{\kappa=1}, & \text{on}\quad \Omega^{\kappa=1}_{\zeta}\\
\tilde{\zeta}^{\kappa=1}_{D\not\to A}, & \text{on}\quad \Omega^{\kappa=1}_{\zeta,D\not\to A}\\
\tilde{\zeta}^{\kappa=0}, & \text{on}\quad \Omega^{\kappa=0}_{\zeta}\\
\tilde{\zeta}^{\kappa=0}_{D\not\to B}, & \text{on}\quad \Omega^{\kappa=0}_{\zeta,D\not\to B}\\
0, & \text{on}\quad \Omega\backslash\Omega_1.
\end{cases}$$

[43] Correspondences are needed because of the possibility of multiple equilibria, which I have not ruled out in my proof. Multiple equilibria are captured mathematically by non-empty intersections of the sets defined above. Numerically, I have found parameterizations of the model in which $\Omega_{\zeta,F\not\to A}\cap\Omega^{\kappa=0}_{\zeta}$ and $\Omega_{\zeta,F\not\to B}\cap\Omega^{\kappa=1}_{\zeta}$ are non-empty. In those cases, I choose assign to (κ,ζ) the values dictated by $\Omega^{\kappa=0}_{\zeta}$ and $\Omega^{\kappa=1}_{\zeta}$.

Proof of Proposition A.3. One can substitute these formulas into (24) and (27) to verify that the equations are solved. It remains to show that the union of the regions defined is equal to the entire state space, i.e.,

$$\Omega_0 \cup \Omega_{0,D\not\to A} \cup \Omega_{0,D\not\to B} \cup \Omega_{0,D\not\to A,B} \cup \Omega_1 = \Omega \tag{89}$$

and

$$\Omega_1^* = \Omega_1, \tag{90}$$

where $\Omega_1^* := \Omega_\zeta \cup \Omega_{\zeta,D\to A} \cup \Omega_{\zeta,D\to B} \cup \Omega_{\zeta,D\to A,B} \cup \Omega_{\zeta,F\to A} \cup \Omega_{\zeta,F\to B} \cup \Omega_\zeta^{\kappa=1} \cup \Omega_\zeta^{\kappa=0} \cup \Omega_{\zeta,D\to A}^{\kappa=1} \cup \Omega_{\zeta,D\to B}^{\kappa=0}$. Statement (89) is trivially true by definition of Ω_1. Furthermore, $\Omega_1^* \subset \Omega_1$ holds trivially, by definition of the sets constituting Ω_1^*.

It remains to prove $\Omega_1^* \supset \Omega_1$. Note that $\zeta > 0$ on Ω_1^*. Thus, $\lambda_F^A + \lambda_F^B = \bar{\lambda}$ on Ω_1^*. The remaining constraints are the shorting constraints of insiders ($\kappa \in [0,1]$), distressed investors ($\lambda_D^A \geq 0, \lambda_D^B \geq 0$), and financiers ($\lambda_F^A \geq 0, \lambda_F^B \geq 0$), a total of 6 constraints. To help characterize these constraints, note the following:

$$\{\tilde{f}_A(\kappa,\zeta) > 0,\, \tilde{d}_A(\kappa,\zeta) > 0\} = \{\lambda_F^A > 0,\, \lambda_D^A > 0\} \tag{91}$$
$$\{\tilde{f}_A(\kappa,\zeta) > 0,\, \tilde{d}_A(\kappa,\zeta) \leq 0\} = \{\lambda_F^A > 0,\, \lambda_D^A = 0\} \tag{92}$$
$$\{\tilde{f}_A(\kappa,\zeta) \leq 0,\, \tilde{d}_A(\kappa,\zeta) > 0\} = \{\lambda_F^A = 0,\, \lambda_D^A > 0\} \tag{93}$$
$$\{\tilde{f}_A(\kappa,\zeta) \leq 0,\, \tilde{d}_A(\kappa,\zeta) \leq 0\} = \{\lambda_F^A = 0,\, \lambda_D^A = 0\}, \tag{94}$$

and, for sector B,

$$\{\tilde{f}_B(\kappa,\zeta) > 0,\, \tilde{d}_B(\kappa,\zeta) > 0\} = \{\lambda_F^B > 0,\, \lambda_D^B > 0\} \tag{95}$$
$$\{\tilde{f}_B(\kappa,\zeta) > 0,\, \tilde{d}_B(\kappa,\zeta) \leq 0\} = \{\lambda_F^B > 0,\, \lambda_D^B = 0\} \tag{96}$$
$$\{\tilde{f}_B(\kappa,\zeta) \leq 0,\, \tilde{d}_B(\kappa,\zeta) > 0\} = \{\lambda_F^B = 0,\, \lambda_D^B > 0\} \tag{97}$$
$$\{\tilde{f}_B(\kappa,\zeta) \leq 0,\, \tilde{d}_B(\kappa,\zeta) \leq 0\} = \{\lambda_F^B = 0,\, \lambda_D^B = 0\}, \tag{98}$$

Importantly, $\{\lambda_F^A = 0,\, \lambda_D^A = 0\} = \{\kappa = 0\}$ and $\{\lambda_F^B = 0,\, \lambda_D^B = 0\} = \{\kappa = 1\}$, by funding market clearing. In addition, $\{\zeta > 0,\, \lambda_F^A = 0,\, \lambda_D^B = 0\} = \{\zeta > 0\, \lambda_F^B = 0,\, \lambda_D^A = 0\} = \varnothing$, by combining equations (28), (23), and (25). Consequently, the sets constituting Ω_1^* are exactly the intersection of $\{\zeta > 0\}$ with the pairwise combinations of the sets in (91)-(94) with the sets in (95)-(98), which are a completely exhaustive set of combinations, i.e., $\{\zeta > 0\} = \Omega_1^*$. Finally note, by the definition of ζ in the statement of Proposition A.3, that $\Omega \backslash \Omega_1 \subset \{\zeta = 0\}$ so that $\Omega_1 \subset \{\zeta > 0\}$. □

A.4 Moral Hazard and Skin-in-the-Game

In the model of Section 2, an insider sells an exogenous fraction ϕ of the capital stock to outsiders (financiers). The insider keeps a fraction of $1-\phi$ of the capital risk on his own balance sheet. In this appendix, I derive this risk-sharing arrangement as the approximate solution to a standard moral hazard problem.

Let the capital stock of a generic insider evolve as follows:

$$dk_{i,t} = k_{i,t}[(\iota_{i,t} - \delta_{i,t})dt + \sigma dZ_t + \hat{\sigma} dW_{i,t}].$$

The new object is $\delta_{i,t}$, which captures hidden diversion. In particular, insiders may divert $\delta k dt$ units of capital to obtain $(1-\phi)\delta k dt$, where ϕ determines the inefficiency from diversion. This may also be thought of as

diverting effort away from capital upkeep.

Insiders hold assets, borrow/lend in risk-free debt markets, and make contractual payments to outsiders. Contractual payments are $-k_{i,t}d\Omega_{i,t}$ per unit of time, since the price of capital is unity in the absence of investment adjustment costs. Since diversion is unobservable, Ω_i must be adapted to the principal's information set, which is generated by (Z, W_i^δ), where $dW_{i,t}^\delta := dW_{i,t} - \delta_{i,t}dt$ is the ex-diversion shock. Contract payments thus take the form $d\Omega_{i,t} = \zeta_{i,t}[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma}dW_{i,t}] + \gamma_{i,t} \cdot dZ_t$ for some processes $(\zeta_{i,t}, \varpi_{i,t}, \gamma_{i,t})$ adapted to (Z, W_i^δ).[44] If both insiders and outsiders may frictionlessly trade claims on the aggregate shock Z, the choice of γ_i is irrelevant. When we assume insiders may not trade claims on Z, i.e., $\theta \equiv 0$, then we are implicitly restricting $\gamma_i = \zeta_i\sigma$. Thus, we use this latter assumption, as it is without loss of generality in the former case. Incorporating these contract payments, insiders' net worth evolution is

$$\begin{aligned} dn_{i,t} = \underbrace{(n_{i,t}r_t - c_{i,t})}_{\text{consumption-savings}}\, dt + \underbrace{(1-\phi)\delta_{i,t}k_{i,t}}_{\text{diversion benefits}}\, dt + \underbrace{k_{i,t}(dR_{i,t} - r_t dt)}_{\text{excess return-on-assets}} \\ - \underbrace{k_{i,t}\zeta_{i,t}\Big[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma}dW_{i,t} + \sigma \cdot dZ_t\Big]}_{\text{contract payments}} + \underbrace{n_{i,t}\theta_{i,t} \cdot (\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}. \end{aligned}$$

This budget constraint has a simple interpretation. Insiders retain a stake $1 - \zeta$ in their asset risks and issue ζ to outsiders. This can be thought of as an equity stake, which has expected excess return ϖ.

Definition 2. Optimal contracts *consist of possible risk exposures and promised payments (i.e., $\zeta_{i,t}$, $\varpi_{i,t}$) that implement no diversion (i.e., $\delta_{i,t} \equiv 0$ for all i, t) and maximize total surplus in the following sense. Taking as given future contracts $\{\zeta_{i,t+s}, \varpi_{i,t+s}\}_{s>0}$, time-$t$ contracts $(\zeta_{i,t}, \varpi_{i,t})$ maximize total instantaneous surplus among contracting parties.*

An important feature of these contracts is that they are short-term, which is captured by the last statement in Definition 2. Contracts are chosen to maximize instantaneous surplus, rather than total long-term surplus, which aids tractability. These short-term contracts would be optimal long-term contracts as well, if agents cannot commit to future contracts and if those future contracts are made in anonymity.

To derive optimal contracts, note that diversion of δk units of capital yields $(1-\phi)\delta k$ in net worth to the insider. On the other hand, the insiders' return-on-assets is reduced by δ, which translates into a $(1-\zeta)\delta k$ lower payoff from inside equity. Consequently, the insider will not divert any capital as long as $\zeta_{i,t} \leq \phi$, which is insiders' incentive-compatibility constraint. In other words, $1 - \phi$ is the minimum skin-in-the-game requirement.

Given competition in the financier sector, ϖ is determined by their marginal utility process, i.e., $\varpi = \hat{\sigma}\hat{\pi} + \sigma \cdot \pi$, where $\hat{\pi}$ is an idiosyncratic risk price (in equilibrium, $\hat{\pi} = (1-\Delta)\hat{\pi}_F$). The important thing about $\hat{\pi}$ is that it is independent of the ζ from this particular contracting problem. We can now write the return on inside equity without diversion,

$$dR_{i,t}^I := r_t dt + \frac{\mu_t - r_t - \zeta_{i,t}\hat{\sigma}\hat{\pi}_{i,t} - \zeta_{i,t}\sigma \cdot \pi_t}{1 - \zeta_{i,t}} dt + \sigma \cdot dZ_t + \hat{\sigma}dW_{i,t},$$

where μ_t is the expected return-on-capital. Insiders' net worth can be re-written in terms of its inside equity position $e_{i,t} := (1-\zeta_{i,t})k_{i,t}$ and the return dR^I as

$$dn_{i,t} = (n_{i,t}r_t - c_{i,t})dt + e_{i,t}(dR_{i,t}^I - r_t dt) + n_{i,t}\theta_{i,t} \cdot (\pi_t dt + dZ_t).$$

[44] Adaptability to W_i^δ implies the weights of $d\Omega_{i,t}$ on $\hat{\sigma}dW_{i,t}$ and $-\delta_{i,t}dt$ must be identical. This weight is $\zeta_{i,t}\hat{\sigma}$. The additional term $\zeta_{i,t}\varpi_{i,t}dt$ allows for time-varying flow payments.

The following facts simplify the analysis: (1) insiders can control their exposure $e_{i,t}$; and (2) financiers' surplus is accounted for by $\varpi_{i,t}$. Consequently, optimal $\zeta_{i,t}$ may be chosen by insiders as they wish, subject to the incentive constraint $\zeta_{i,t} \leq \phi$. Although I assume $\zeta_{i,t} = \phi$ in the main text, I detail the actual solution below which helps understand how good the assumption $\zeta_{i,t} = \phi$ is.

Since $\zeta_{i,t}$ only affects $\mathbb{E}_t[dR^I_{i,t}]$ and not $dR^I_{i,t} - \mathbb{E}_t[dR^I_{i,t}]$, optimal $\zeta_{i,t}$ is chosen to maximize $\mathbb{E}_t[dR^I_{i,t}] - r_t dt$, i.e.,

$$\max_{\zeta \in [0,\phi]} \left\{ \frac{\mu - r - \zeta \hat{\sigma}\hat{\pi} - \zeta \sigma \cdot \pi}{1 - \zeta} \right\}.$$

It is optimal to set $\zeta = \phi$ when $\mu - r - \hat{\sigma}\hat{\pi} - \sigma \cdot \pi > 0$ and to otherwise set ζ such that $\mu - r - \hat{\sigma}\hat{\pi} - \sigma \cdot \pi = 0$. Doing this requires knowledge of the equilibrium risk prices, which depends on the market structure (e.g., whether or not θ is constrained or unconstrained). As an example, use the expressions from Proposition 2.4 for sector A, with ζ in place of ϕ, to get

$$\mu - r - \hat{\sigma}\hat{\pi} - \sigma \cdot \pi = (1 - \zeta)\kappa \hat{\sigma}^2 \Big[\frac{1 - \zeta}{(1 - \eta)\alpha} - \frac{\zeta (1 - \Delta)^2}{\eta} \Big].$$

The solution is

$$\zeta_{i,t} = \min \Big(\phi, \ \Big[1 + (1 - \Delta)^2 \frac{\alpha_t (1 - \eta_t)}{\eta_t} \Big]^{-1} \Big). \tag{99}$$

Notice that $\zeta_{i,t} = \phi$ is optimal across the state space if $\Delta = 1$, which is the case commonly studied in the literature (e.g., Di Tella (2017)). When financiers are imperfectly diversified, $\zeta_{i,t} < \phi$ is possible for very low values of η_t, because financiers' required rate of return diverges to infinity. However, notice that in the steady-state equilibrium of Proposition 2.5, wealth shares are such that maximal issuance $\zeta_{i,t} = \phi$ is optimal. Furthermore, as Δ increases, the possibility of unconstrained risk-sharing shrinks. Notice that $\zeta_{i,t} < \phi$ when $\eta_t < \eta^*_t$, where

$$\eta^*_t := \frac{\phi \alpha_t (1 - \Delta)^2}{1 - \phi + \phi \alpha_t (1 - \Delta)^2},$$

which shrinks to 0 at a quadratic rate as $\Delta \to 1$, i.e.,

$$\frac{d \log \eta^*}{d \log (1 - \Delta)} = \frac{2[1 - \phi + \phi \alpha (1 - \Delta)^2 - \phi \alpha (1 - \Delta)^2]}{1 - \phi + \phi \alpha (1 - \Delta)^2} = 2(1 - \eta^*).$$

Hence, for relatively high values of Δ such as those considered in the quantitative section, the assumption of $\zeta_{i,t} = \phi$ is innocuous.

A.5 Endogenous Credit Standards

By smoothing out the moral hazard problem of Appendix A.4, we can address the question of how diversification affects credit standards. To make these concepts precise, consider a more general hidden diversion technology. Suppose diverting $\delta k dt$ units of capital yields an income flow of $h(\delta) k dt$ to the insider, where $h(\cdot)$ satisfies the following.

Assumption 4 (Diversion Benefit)**.** *Assume the function $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is twice-differentiable with the following properties: $h(0) = 0$, $h'(\delta) \leq 1$ for all δ, $h''(\delta) \leq 0$ for all δ, and $h'(+\infty) = 0$.*

In Appendix A.4, we had assumed a linear diversion technology $h(\delta) = (1 - \phi)\delta$, which satisfies Assumption

4. In this more general formulation, insiders' net worth evolves as

$$dn_{i,t} = \underbrace{(n_{i,t}r_t - c_{i,t})}_{\text{consumption-savings}}\,dt + \underbrace{h(\delta_{i,t})k_{i,t}}_{\text{diversion benefits}}\,dt + \underbrace{k_{i,t}(dR_{i,t} - r_t dt)}_{\text{excess return-on-assets}}$$
$$\underbrace{- k_{i,t}\zeta_{i,t}\Big[(\varpi_{i,t} - \delta_{i,t})dt + \hat{\sigma}dW_{i,t} + \sigma \cdot dZ_t\Big]}_{\text{contract payments}} + \underbrace{n_{i,t}\theta_{i,t} \cdot (\pi_t dt + dZ_t)}_{\text{aggregate risk hedging}}.$$

As before, $(\zeta_{i,t}, \varpi_{i,t})$ characterize contract payments.

With this general specification of h, it may not be desirable to implement zero diversion. For instance, if $h'(0) = 1$, implementing no diversion requires insiders to keep 100% skin-in-the-game, i.e., $\zeta_{i,t} \equiv 0$. But such a contract may impose too much risk onto insiders' balance sheets. Thus, "optimal contracts" in this setting remove from Definition 2 the requirement $\delta_{i,t} \equiv 0$.

To solve this contracting problem, we may repeat a similar analysis to Appendix A.4. Given a skin-in-the-game ζ, optimal diversion maximizes $h(\delta) - (1-\zeta)\delta$. Thus, $h'(\delta) \leq 1 - \zeta$. Since h' is weakly decreasing, optimal diversion is a weakly increasing function $f(\zeta)$. Next, optimal payments are $\varpi = f(\zeta) + \hat{\sigma}\hat{\pi} + \sigma \cdot \pi$, which now compensates financiers for the possibility that $\delta > 0$. As before, this means that it suffices to consider insiders' surplus, which is governed solely by their expected return on inside equity, i.e.,

$$\max_{\zeta \in [0,1]} \left\{ \frac{\mu - r - f(\zeta) + h(f(\zeta)) - \zeta\hat{\sigma}\hat{\pi} - \zeta\sigma \cdot \pi}{1 - \zeta} \right\}.$$

Supposing f is differentiable and that optimal diversion is positive, $\delta > 0$, we have the first-order optimality condition

$$\mu - r - \hat{\sigma}\hat{\pi} - \sigma \cdot \pi = \zeta(1-\zeta)f'(\zeta) + f(\zeta) - h(f(\zeta)).$$

Modify the equilibrium expressions from Proposition 2.4 for sector A by putting ζ in place of ϕ and accounting for diversion benefits and costs:

$$\mu - r - f(\zeta) + h(f(\zeta)) - \sigma \cdot \pi = (1-\zeta)\hat{\sigma}\frac{\kappa(1-\zeta)\hat{\sigma}}{(1-\eta)\alpha} + \zeta(1-\Delta)\hat{\sigma}\frac{\kappa\zeta(1-\Delta)\hat{\sigma}}{\eta}.$$

Substitute the optimality condition for ζ to get the equilibrium condition:

$$\underbrace{(1-\zeta_{i,t})\zeta_{i,t}f'(\zeta_{i,t})}_{\text{marginal cost of issuance}} = \underbrace{\kappa_t(1-\zeta_{i,t})\hat{\sigma}^2\Big[\frac{1-\zeta_{i,t}}{(1-\eta_t)\alpha_t} - \frac{\zeta_{i,t}(1-\Delta)^2}{\eta_t}\Big]}_{\text{marginal benefit of issuance}}. \tag{100}$$

After dividing both sides by $1-\zeta$, the left-hand-side of (100) is strictly increasing in ζ, whereas the right-hand-side is strictly decreasing. Thus, there is a uniquely optimal skin-in-the-game in equilibrium. This equilibrium equates the "marginal benefit of issuance," comprised by diversification benefits from offloading risk, to the "marginal cost of issuance," defined by the marginal increase in diversion, net of the marginal private diversion benefits.[45]

With this smooth moral hazard setup, we can analyze the effects of diversification. From (100), higher Δ reduces optimal skin-in-the-game $1-\zeta$, which is a generalization of the results in Appendix A.4. In other words, insiders issue more securities to better-diversified outsiders. But interestingly, this now comes with a cost. Higher ζ increases equilibrium diversion $f(\zeta)$ and thus deadweight losses $f(\zeta) - h(f(\zeta))$. This result is

[45] That is, the marginal cost is $(1-\zeta)\frac{d}{d\zeta}[f(\zeta) - h(f(\zeta))] = (1-\zeta)[f'(\zeta) - (1-\zeta)f'(\zeta)] = \zeta(1-\zeta)f'(\zeta)$.

analogous to the story that "securitization leads to lax screening" as in Keys, Mukherjee, Seru and Vig (2010).

A.6 Micro-founding the Leverage Constraint

In section 3, I have assumed the leverage constraint (22) is defined an exogenous and constant maximum $\bar{\lambda}$. Several macro-finance papers derive a bank leverage constraint from an underlying agency friction, e.g., the simple limited commitment problem in Gertler and Kiyotaki (2010). In that case, the constraint is not a constant level, but rather a function of the economic state and model parameters.

What is particularly important for my model is how diversification could affect the leverage constraint. Does better diversification relax the leverage constraint such that it would never bind in equilibrium? In this section, I use a simple agency problem similar to Gertler and Kiyotaki (2010) to derive the leverage constraint and show that it is insensitive to diversification improvements. Thus, the constraint is indeed more likely to bind as diversification improves.

We consider the model of Section 3 with symmetric discount rates (ρ), no OLG ($\delta = 0$), and a single productive sector ($G_A = G, G_B = -\infty$ such that $\kappa = 1$; I drop all A subscripts accordingly). Suppose financiers can abscond with a fraction $\gamma \in (0,1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, financiers would have net worth $\tilde{n}^F_{i,t} := \gamma\lambda_t n^F_{i,t}$. I consider two cases. After diversion, financiers move to another location and either (i) set up a new financial intermediary anonymously; or (ii) retire to become insiders in the single productive sector A. A fact I will use in both cases is the following: one can show that the value functions of financiers and insiders are given by $\log(n^F_{i,t}) + \xi_{F,t}$ and $\log(n^I_{i,t}) + \xi_{I,t}$, respectively.

In case (i), diversion delivers utility $\log(\tilde{n}^F_{i,t}) + \xi_{F,t}$, which must be ruled out by the incentive constraint $\log(\tilde{n}^F_{i,t}) + \xi_{F,t} \leq \log(n^F_{i,t}) + \xi_{F,t}$. As a result,

$$\lambda_t \leq \gamma^{-1}$$

is required. Therefore, the leverage limit $\bar{\lambda} \equiv \gamma^{-1}$ is completely independent of diversification.

In case (ii), the same analysis delivers the incentive constraint

$$\lambda_t \leq \gamma^{-1} \exp(\xi_{F,t} - \xi_{I,t}).$$

Here, the leverage limit $\bar{\lambda} \equiv \gamma^{-1}\exp(\xi_F - \xi_I)$ plausibly depends on diversification through the relative investment opportunities of financiers and insiders, captured by $\xi_F - \xi_I$. However, in a steady state in which the leverage constraint is conjectured to not bind for all other financiers (besides the one whose agency problem we are focusing on), simple calculations show $\xi_{F,\infty} - \xi_{I,\infty} = 0$. Intuitively, the long-run wealth distribution adjusts such that all agents earn the same idiosyncratic risk prices, which are the key determinants of ξ_F and ξ_I. Again, the leverage constraint is independent of diversification.

A.7 Welfare

Under construction.

A.8 Impulse Responses

Define the IRF of a stationary variable Y by

$$\mathcal{I}[Y](t,x;\Delta) := \mathbb{E}[Y_{\tau+t} - Y_{\tau-} \mid X_{\tau-} = x], \quad t \geq 0, \tag{101}$$

where X_t is the vector of state variables, i.e., $X_t = (\alpha_t, \eta_t)$ in Section 2, and Δ is the variable receiving an unanticipated shock at time τ, i.e., $\Delta_\tau \neq \Delta_{\tau-}$. Equation (101) can be decomposed into the sum of an "impact response" $\mathbb{E}[Y_\tau - Y_{\tau-} \mid X_{\tau-} = x]$ and a "transition path" $\mathbb{E}[Y_{\tau+t} - Y_\tau \mid X_\tau = x']$.

In the baseline model of Section 2, we first considered one-time unanticipated shocks to (Δ_A, Δ_B). An important simplifying property of this model is that these shocks do not generate any impact response to the state variables in the model, as stated in Lemma 2.8. Thus, $\mathcal{I}[X](t, x; \Delta) = \mathbb{E}[X_{\tau+t} - X_\tau \mid X_\tau = x]$ for this type of shock. Figure 7 in the main text does this IRF analysis for a one-time shock to Δ_A. Figure 8 repeats the analysis for a gradual increase in Δ_A, which may be interpreted in three equivalent ways – fully unanticipated, fully anticipated, or partially anticipated – in the sense of Lemma 2.9.

Proof of Lemma 2.8. We want to show that the impact responses $\mathbb{E}[\eta_\tau - \eta_{\tau-} \mid X_{\tau-} = x] = \mathbb{E}[\alpha_\tau - \alpha_{\tau-} \mid X_{\tau-} = x] = 0$ in response to a shock to Δ_A at time τ. A similar analysis holds for state variable α and for a shock to Δ_B. Note that aggregated net worths can be written as

$$\begin{aligned}
N_{F,t} &= [\kappa_t \phi_A + (1-\kappa_t)\phi_B]K_t + M_{F,t} + p_t \tilde{\theta}_{F,t} N_{F,t} \\
N_{A,t} &= \kappa_t (1-\phi_A) K_t + M_{A,t} + p_t \tilde{\theta}_{A,t} N_{A,t} \\
N_{B,t} &= (1-\kappa_t)(1-\phi_B) K_t + M_{B,t} + p_t \tilde{\theta}_{B,t} N_{B,t},
\end{aligned}$$

where $M_{z,t}$ are the riskless assets, $\tilde{\theta}_{z,t} N_{z,t}$ are the hedging securities (Arrow securities on the aggregate shock) held, and p_t the price of these securities. Bond market clearing is that $M_{F,t} + M_{A,t} + M_{B,t} = 0$. Notice that equity market clearing is already imposed. Since the Arrow securities pay an excess return, their price is zero, i.e., $p_t = 0$ for all t. We can thus rewrite the net worth equations immediately prior to the shock as

$$\begin{aligned}
N_{F,\tau-} &= [\kappa_{\tau-} \phi_A + (1-\kappa_{\tau-})\phi_B]K_{\tau-} - M_{A,\tau-} - M_{B,\tau-} \\
N_{A,\tau-} &= \kappa_{\tau-} (1-\phi_A) K_{\tau-} + M_{A,\tau-} \\
N_{B,\tau-} &= (1-\kappa_{\tau-})(1-\phi_B) K_{\tau-} + M_{B,\tau-}.
\end{aligned}$$

Upon the shock, the variables $K, \kappa, 1-\kappa, M_A, M_B$ cannot jump. Indeed, a jump in K cannot be consistent with goods market clearing, whereas M_A, M_B represent trading positions (asset holdings) and are assumed fixed at the time of the shock. Furthermore, the Arrow securities are settled every dt time periods, so they have no continuation value that could possibly jump. Then, after the shock,

$$\begin{aligned}
N_{F,\tau} &= [\kappa_{\tau-} \phi_A + (1-\kappa_{\tau-})\phi_B]K_{\tau-} - M_{A,\tau-} - M_{B,\tau-} \\
N_{A,\tau} &= \kappa_{\tau-} (1-\phi_A) K_{\tau-} + M_{A,\tau-} \\
N_{B,\tau} &= (1-\kappa_{\tau-})(1-\phi_B) K_{\tau-} + M_{B,\tau-}.
\end{aligned}$$

Therefore, $N_{z,\tau} = N_{z,\tau-}$ for all agents z. As a result, $\eta_\tau - \eta_{\tau-} = \alpha_\tau - \alpha_{\tau-} = 0$. □

Proof of Lemma 2.9. Part (i) is a special case of part (ii) for $t \neq \tau$. For $t = \tau$, we make use of Proposition 2.8. Part (ii) follows from the fact that optimal choices of all log utility agents in the model are independent of the Itô processes driving expected returns and volatilities, see Appendix A.1. Hence, Proposition 3.1 still holds even in this more general case. The equilibrium of Proposition 2.4 follows as a special case of Proposition 3.1, as before. □

A.9 Other Shocks

This section presents proofs and details for the results of Section 4.

Proof of Proposition 4.1. Using the notation of Lemma **??**, we have partial derivatives (for any parameter values)

$$C_{\alpha,\phi_A} = -\frac{\kappa_\infty \alpha_\infty \hat{\sigma}_A}{\kappa_\infty(1-\phi_A)\hat{\sigma}_A + (1-\kappa_\infty)(1-\phi_B)\hat{\sigma}_B}$$
$$C_{\eta,\phi_A} = \hat{\pi}_\infty^{-2}\Big[\frac{1-\eta_\infty}{\eta_\infty}\phi_A\kappa_\infty^2(1-\Delta_A)^2\hat{\sigma}_A^2 + \eta_\infty\hat{\pi}_\infty\kappa_\infty\hat{\sigma}_A\Big]$$
$$C_{\kappa,\phi_A} = -\frac{2\kappa_\infty}{M_A+M_B}\Big[\frac{\phi_A(1-\Delta_A)^2\hat{\sigma}_A^2}{\eta_\infty} - \frac{(1-\phi_A)\hat{\sigma}_A^2}{\alpha_\infty(1-\eta_\infty)}\Big].$$

Imposing symmetric parameters, we have $\alpha_\infty = \kappa_\infty = 1/2$ so that $D_{\eta,\kappa} = D_{\kappa,\eta} = 0$ and $D_{\alpha,\kappa} = 1$ and $D_{\kappa,\alpha} = \frac{2}{M}(1-\eta_\infty)^{-1}(1-\phi)^2\hat{\sigma}^2 < 1$. Then, use expressions (77)-(78) to verify that

$$\frac{d\kappa_\infty}{d\phi_A} = [1-\sqrt{2}(1-\Delta_A)]\hat{\pi}_\infty\hat{\sigma}_A$$
$$\frac{d\eta_\infty}{d\phi_A} = (1-D_{\alpha,\kappa})C_{\eta,\phi_A}.$$

Note that $\frac{d\eta_\infty}{d\phi_A} > 0$ unambiguously, while $\frac{d\kappa_\infty}{d\phi_A} > 0$ if and only if $\Delta_A > 1 - 1/\sqrt{2}$. □

Proof of Proposition 4.2. Repeat the steps of Proposition 2.4 with λ^z replaced by $(1-\phi_F)\lambda^z$, with funding market clearing replaced by $(1-\phi_F)\int_{i-\Delta_z}^{i}\Delta_z^{-1}[\lambda_{j,t}^z n_{j,t}^F dj = \phi_z k_{i,t}^z$, and with aggregate risk market clearing replaced by $\int_0^1[\theta_{i,t}^A n_{i,t}^A + \theta_{i,t}^B n_{i,t}^B + \theta_{i,t}^F n_{i,t}^F]di = (1-\phi_F)\int_0^1(\lambda_{i,t}^A\sigma_A + \lambda_{i,t}^B\sigma_B)n_{i,t}^F di$. □

Proof of Proposition 4.3. Under construction. □

Proof of Proposition 4.4. Under construction. □

Proof of Proposition 4.5. Under construction. □

B Results on the Brownian Cylinder W

B.1 Aggregate risk along investment arcs

Proof of Lemma 2.2. To examine the degree of aggregate risk in this economy, consider investing one unit of consumption, divided equally amongst each market in $[\frac{1-\Delta}{2}, \frac{1+\Delta}{2}]$ (the fact that it is centered at $1/2$ is without loss of generality, by symmetry). This results in:

$$\begin{aligned}
\text{Var}_t\Big(\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \Delta^{-1} dW_{i,t} di\Big) &= \Delta^{-2}\text{Cov}_t\Big(\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} dW_{i,t} di, \int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} dW_{j,t} dj\Big) \\
&= \Delta^{-2}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \text{Cov}_t(dW_{i,t}, dW_{j,t}) didj \\
&= \Big(1 - 6\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}}\int_{\frac{1-\Delta}{2}}^{\frac{1+\Delta}{2}} \Delta^{-2}|i-j|(1-|i-j|) didj\Big)dt \\
&= \Big(1 - 6\int_0^1\int_0^1 \Delta|x-y|(1-\Delta|x-y|)dxdy\Big)dt \\
&= \Big(1 - 6\int_{-1}^1 (1-|u|)(1-\Delta|u|)\Delta|u| du\Big)dt \\
&= (1-\Delta)^2 dt.
\end{aligned}$$

In the third line, I have substituted the covariance and distance metric: $\text{Cov}_t(dW_{i,t}, dW_{j,t}) = 1 - 6\min(|i-j|, 1-|i-j|)(1-\min(|i-j|(1-|i-j|)) = 1 - 6|i-j|(1-|i-j|)$. In the fourth line, I have performed the change-of-variables $i = \frac{1-\Delta}{2} + \Delta x$ and $j = \frac{1-\Delta}{2} + \Delta y$. In the fifth line, I have substituted $u = x - y$ and used the fact that if X and Y are independent uniform random variables, then $X - Y$ has the triangular distribution. Given this formula, we may take $\Delta \to 1$ to see that $\text{Var}_t(\int_0^1 dW_{i,t} di) = 0$. As this expectation is zero, this shows that $\int_0^1 dW_{i,t} di = 0$ almost-surely. □

B.2 Existence of W

One may ask whether or not such a stochastic process $W := \{W_{i,t} : i \in [0,1], t \geq 0\}$ exists on any probability space. In other words, are the properties assumed above mutually consistent? Below, I prove that such shocks exist by an implicit method, using the theory of Gaussian processes.

This relates to the class of Gaussian random fields that are used to model forward rates in Kennedy (1994), which to my knowledge is the first use of such processes in financial economics. The key property aiding the analysis of that paper, as in this paper, is the independent increments property of the random field in the "time" direction. Santa-Clara and Sornette (2001) study a similar stochastic process, which they call "string" shocks. They obtain these shocks using the theory of stochastic partial differential equations (SPDEs), although I prove existence in a different way. That said, the W process is not a special case of the class of processes they consider. Furthermore, my existence proof is general enough to apply analogously to their entire class of processes.

First, I build a particular Gaussian process. Second, I show that this stochastic process has the desired properties. Given the construction, which posits the covariance in the t-direction and i-direction as multiplicatively separable, and the property that the process acts as a continuum of Wiener processes in the t-direction, W is thus an example of a cylindrical Wiener process (see a reference on SDEs in infinite dimensions, e.g., Da Prato and Zabczyk (2014)).

Proof of Lemma 2.1. The existence of a mean-zero Gaussian process having covariance function

$$V((i,s),(j,t)) = \Big[1 - 6\text{dist}(i,j)(1-\text{dist}(i,j))\Big] \times \min(s,t)$$

is guaranteed if and only if V is symmetric and positive semi-definite (see any reference on Gaussian processes, e.g., proposition I.24.2 in Rogers and Williams (2000)). Clearly, V is symmetric. To check positive semi-definiteness, construct the Gram matrix: let $i_1,\ldots,i_N \in [0,1]$ and $t_1,\ldots,t_N \in \mathbb{R}_+$, and define the matrix G by

$$G := [V((i_m,t_m),(i_n,t_n))]_{m,n\in\{1,\ldots,N\}}.$$

We need to show that G is positive semi-definite. To do this, define the "univariate" covariance functions $v_1(i,j) := V((i,1),(j,1))$ and $v_2(s,t) := V((0,s),(0,t))$, and the associated Gram matrices

$$G_1 := [v_1(i_m,i_n)]_{m,n\in\{1,\ldots,N\}} \quad \text{and} \quad G_2 := [v_2(t_m,t_n)]_{m,n\in\{1,\ldots,N\}}.$$

Notice that

$$G = G_1 \circ G_2,$$

where $\circ$ denotes the Schur product (element-wise multiplication). By the Schur product theorem, it suffices to show that G_1 and G_2 are both positive semi-definite, because then so is G.

Consider a standard Brownian bridge process $\{W_i^\circ : i \in [0,1]\}$ and define the process

$$B_i := \sqrt{12}\Big[W_i^\circ - \int_0^1 W_j^\circ dj\Big].$$

Note that $\mathbb{E}B_i = 0$ for all i and

$$\begin{aligned}
\mathbb{E}B_iB_j &= 12\mathbb{E}[(W_i^\circ - \int_0^1 W_k^\circ dk)(W_j^\circ - \int_0^1 W_k^\circ dk)] \\
&= 12\Big[\mathbb{E}W_i^\circ W_j^\circ + \mathbb{E}\int_0^1\int_0^1 W_k^\circ W_l^\circ dkdl - \mathbb{E}\int_0^1 W_i^\circ W_k^\circ dk - \mathbb{E}\int_0^1 W_j^\circ W_k^\circ dk\Big] \\
&= 12\Big[\min(i,j) - ij + \int_0^1\int_0^1 [\min(k,l)-kl]dkdl - \int_0^1[\min(i,k)-ik]dk - \int_0^1[\min(j,k)-jk]dk\Big] \\
&= 12\Big[\min(i,j) - ij + \int_0^1 \frac{l(1-l)}{2}dl - \frac{i(1-i)}{2} - \frac{j(1-j)}{2}\Big] \\
&= 1 - 6|i-j|(1-|i-j|).
\end{aligned}$$

In the third and fourth equality, I have used the Brownian bridge covariance function to compute $\mathbb{E}W_i^\circ W_j^\circ = \min(i,j) - ij$, as well as the integral

$$\begin{aligned}
\int_0^1 [\min(i,j)-ij]dj &= \int_0^i j(1-i)dj + \int_i^1 i(1-j)dj \\
&= \frac{1}{2}i^2(1-i) + \frac{1}{2}i(1-i)^2 \\
&= \frac{i(1-i)}{2}.
\end{aligned} \tag{102}$$

Therefore, v_1 is the covariance function for B. As a valid covariance function, we immediately conclude that G_1 is positive semi-definite. Finally, v_2 is the covariance function of standard Brownian motion, so the matrix

G_2 is positive semi-definite.

Thus, define W to be a Gaussian process with covariance function V. We want to show that W has the desired properties from Assumption 1 of the text: (1) at each location, W acts as a Brownian motion; (2) dW has the correct cross-sectional correlations; (3) W has a path-continuous version.

First, fixing i, the time-series process $W^{(i)} := \{W_{i,t} : t \geq 0\}$ is a standard Brownian motion. Indeed, $\mathbb{E}[W_{i,0}^2] = 0$ implies $W_{i,0} = 0$ almost-surely. Since $W^{(i)}$ is a centered Gaussian process with $V((i,s),(i,t)) = \min(s,t)$, it has the same probability law as a standard Brownian motion. Having the same probability law, it is well-known that $W^{(i)}$ can be chosen to be path-continuous. Independent increments can be established as follows. Using the covariance function and the Normal distribution, we have $\mathbb{E}[(W_{i,t_2} - W_{i,t_1})(W_{i,t_1} - W_{i,t_0})] = 0$ for $t_2 \geq t_1 \geq t_0 \geq 0$. Orthogonality plus joint Normality implies independence of $W_{i,t_2} - W_{i,t_1}$ from $W_{i,t_1} - W_{i,t_0}$.

Second, the increments to $W^{(i)}$ and $W^{(j)}$ have the desired pairwise correlations. Indeed, using the covariance function V, we have

$$\frac{1}{s}\mathbb{E}[(W_{i,t+s} - W_{i,t})(W_{j,t+s} - W_{j,t})] = 1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j)).$$

As $s > 0$ is arbitrary, and using the Markov property of Brownian motion, we have that

$$\text{corr}(dW_{i,t}, dW_{j,t} \mid \mathcal{F}_t) = 1 - 6\text{dist}(i,j)(1 - \text{dist}(i,j)).$$

Third, we can use the Kolmogorov-Chentsov continuity criterion (see any reference on Gaussian processes, e.g., theorem I.25.2 in Rogers and Williams (2000)) to show that W has a version with continuous sample paths.[46] To do this, we may fix an arbitrary $T > 0$ and show that there exist $C > 0$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ (which may all depend on T) such that

$$\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} \leq C \times \text{dist}((i,s),(j,t))^{2(1+\varepsilon_2)}, \quad \forall s,t \leq T, \tag{103}$$

where $\text{dist}(\cdot,\cdot)$ is Euclidean distance in $\mathbb{C}_1^\circ \times \mathbb{R}$, where $\mathbb{C}_1^\circ$ is the circle of circumference one.[47] In particular,

$$\text{dist}((i,s),(j,t)) := \sqrt{|s-t|^2 + |i-j|^2(1-|i-j|)^2}.$$

Assume $s > t$ (the opposite case follows symmetrically). Set $\varepsilon_2 > 0$ arbitrarily, and set $\varepsilon_1 = 4(1+\varepsilon_2)$. Then, because W is Gaussian, there exists a constant M such that

$$\begin{aligned}\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} &= \mathbb{E}|W_{i,s} - W_{j,t}|^{4(1+\varepsilon_2)} \\ &= M\mathbb{E}[|W_{i,s} - W_{j,t}|^2]^{2(1+\varepsilon_2)}.\end{aligned}$$

[46] One might think we could prove continuity by appealing to the fact that $\{W_{i,t} : i \in [0,1]\}$ is a translated, scaled Brownian bridge for each t, and $\{W_{i,t} : t \geq 0\}$ is a Brownian motion for each i. Thus, we could construct continuous versions of each of these at the rational indexes, and use a density argument to construct a continuous W in the limit. The problem with this approach is that we don't know that the limiting process has the desired distributional properties.

[47] This is a slight generalization of the conventional Kolmogorov-Chentsov theorem, in which the index set is not $\mathbb{R}^2$. But the exact same condition applies.

Compute, using the triangle inequality, the covariance function, and the assumption that $t < s < T$:

$$\begin{aligned}
\mathbb{E}|W_{i,s} - W_{j,t}|^2 &= V((i,s),(i,s)) + V((j,t),(j,t)) - 2V((i,s),(j,t)) \\
&\leq |V((i,s),(i,s)) - V((i,s),(j,t))| + |V((j,t),(j,t)) - V((i,s),(j,t))| \\
&\leq 2|V((i,s),(j,t)) - V((j,t),(j,t))| + |s-t| \\
&= 2|\min(s,t)(1 - 6|i-j|(1-|i-j|)) - t| + |s-t| \\
&= 12t|i-j|(1-|i-j|) + |s-t| \\
&\leq 24\max(T,1/12)[|s-t|^2 + |i-j|^2(1-|i-j|)^2]^{1/2} \\
&= 24\max(T,1/12)\text{dist}((i,s),(j,t))
\end{aligned}$$

Consequently,

$$\mathbb{E}|W_{i,s} - W_{j,t}|^{\varepsilon_1} \leq M(24\max(T,1/12))^{2(1+\varepsilon_2)}\text{dist}((i,s),(j,t))^{2(1+\varepsilon_2)},$$

which is condition (103) with $C = M(24\max(T,1/12))^{2(1+\varepsilon_2)}$. □

B.3 Simulating W

Given that Lemma 2.1 only implicitly defines W, one might wonder how such a process could be simulated. In this section, I provide a method for simulating W. I do not show that this construction satisfies all the technical measurability requirements, but I do show that the simulated process has all of the relevant properties if those technical requirements are satisfied.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space satisfying the usual conditions, on which $W^* := \{W_t^* : t \geq 0\}$ is a standard Brownian motion adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$. In addition, let $\hat{B}_t := \{\hat{B}_{i,t} : i \in [0,1]\}$ be a standard Brownian bridge for every $t > 0$. Then, let $\{u_t\}$ be iid Uniform$[0,1]$ random variables, independent of W^* and $\hat{B}$, and define

$$B_{i,t} := \begin{cases} \hat{B}_{u_t+i,t}, & \text{if } i \in [0, 1-u_t] \\ \hat{B}_{u_t+i-1,t}, & \text{if } i \in (1-u_t, 1]. \end{cases}$$

Thus, B_t is essentially a Brownian bridge for every t, but with a random initial index on the circle, u_t. In fact, $B_{i,t} - B_{0,t}$ is a standard Brownian bridge.[48] Assume that the sequence of Brownian bridges $B := \{B_t : t > 0\}$ is progressively measurable. Define the process

$$W_{i,t} := \sqrt{12}\int_0^t \left(B_{i,s} - \int_0^1 B_{j,s}dj\right)dW_s^* \tag{104}$$

so that the increment is $dW_{i,t} = \sqrt{12}\big(B_{i,t} - \int_0^1 B_{j,t}dj\big)dW_t^*$. I will show that $W_{i,t}$ satisfies Assumption 1. In that case, simulating the process requires an approximation to a single aggregate Brownian motion $\{W_t^* : t = 0, dt, 2dt, \dots\}$, a sequence of iid uniform random variables $\{u_t : t = 0, dt, 2dt, \dots\}$, and a Brownian bridge drawn independently at each time step, $\{\hat{B}_{i,t} : i = 0, di, 2di, \dots, 1, \quad t = 0, dt, 2dt, \dots\}$.[49]

[48] One can check this by simply by verifying that this process has the correct covariance function, i.e., for $i \leq j$, $\text{Cov}(B_{i,t} - B_{0,t}, B_{j,t} - B_{0,t}) = i(1-j)$.

[49] This last step can even be relaxed for speed. Due to the iid nature of u_t, one may draw a single Brownian bridge only and achieve the desired properties.

First, I verify property (i) that $W_{i,t}$ is a standard Brownian motion for each i. As a stochastic integral, $W_{i,t}$ is a continuous local martingale in t. Next, the process $W_{i,t}^2 - t$ is a local martingale, since

$$\begin{aligned}
\mathbb{E}_t[W_{i,t+T}^2] &= 12\mathbb{E}_t\Big[\Big(\int_0^{t+T}\Big(B_{i,s} - \int_0^1 B_{j,s}dj\Big)dW_s^*\Big)^2\Big] \\
&= W_{i,t}^2 + 12\mathbb{E}\Big[\int_t^{t+T}\Big(B_{i,s} - \int_0^1 B_{j,s}dj\Big)^2 ds\Big] \\
&= W_{i,t}^2 + 12\int_t^{t+T}\Big[\mathbb{E}(B_{i,s} - B_{0,s})^2 + \mathbb{E}\Big(\int_0^1 (B_{j,s} - B_{0,s})dj\Big)^2 - 2\mathbb{E}(B_{i,s} - B_{0,s})\int_0^1 (B_{j,s} - B_{0,s})dj\Big]ds \\
&= W_{i,t}^2 + 12\int_t^{t+T}\Big[i(1-i) + \int_0^1 \frac{j(1-j)}{2}dj - i(1-i)\Big]ds \\
&= W_{i,t}^2 + t + (T-t).
\end{aligned}$$

Note that $\mathbb{E}_t$ denotes conditional expectation using the information set at time t, not including the information on u_t. In the second equality, I have used the independence of $\{B_{i,t+s} : i \in [0,1], s > 0\}$ and $\{W_{t+s}^* : s > 0\}$ from $\mathcal{F}_t$ as well as Itô's isometry. In the third equality, I have used Fubini's theorem. In the fourth equality, I have used the fact that $\{B_{i,t} - B_{0,t} : i \in [0,1]\}$ is a standard Brownian bridge, combined with the calculation in (102). By Lévy's characterization of Brownian motion, $W_{i,t}$ is a standard Brownian motion for each i.

To prove property (ii) of Assumption 1, recall the quadratic covariation formula: for any two continuous local martingales M and N, $\mathbb{E}_t M_{t+T}N_{t+T} - M_tN_t = \mathbb{E}_t([M,N]_{t+T}) - [M,N]_t$. Using $M_t = W_{i,t}$ and $N_t = W_{j,t}$, and noting that these local martingales have the Markov property (by virtue of being Brownian motions), we have an unconditional version of this covariation formula:

$$\mathbb{E}_t[(W_{i,t+T} - W_{i,t})(W_{j,t+T} - W_{j,t})] = \mathbb{E}[(W_{i,t+T} - W_{i,t})(W_{j,t+T} - W_{j,t})] = \mathbb{E}([W_i, W_j]_{t+T}) - \mathbb{E}([W_i, W_j]_t).$$

The quadratic covariation is

$$[W_i, W_j]_t = 12\int_0^t \Big(B_{i,s} - \int_0^1 B_{k,s}dk\Big)\Big(B_{j,s} - \int_0^1 B_{k',s}dk'\Big)ds,$$

so that, using similar moments for the Brownian bridge as above, we have for $i < j$,

$$\begin{aligned}
&\mathbb{E}_t[(W_{i,t+T} - W_{i,t})(W_{j,t+T} - W_{j,t})] \\
&= 12\mathbb{E}\int_t^{t+T}\Big(B_{i,s} - \int_0^1 B_{k,s}dk\Big)\Big(B_{j,s} - \int_0^1 B_{k',s}dk'\Big)ds \\
&= 12\int_t^{t+T}\Big[\mathbb{E}B_{i,s}B_{j,s} + \mathbb{E}\Big(\int_0^1 B_{k,s}dk\Big)^2 - \mathbb{E}B_{i,s}\int_0^1 B_{k,s}dk - \mathbb{E}B_{j,s}\int_0^1 B_{k,s}dk\Big]ds \\
&= 12\Big[i(1-j) + \frac{1}{12} - \frac{i(1-i)}{2} - \frac{j(1-j)}{2}\Big]T = [1 - 6(j-i)(1-(j-i))]T.
\end{aligned}$$

This shows that

$$\frac{1}{dt}\mathbb{E}_t[dW_{i,t}dW_{j,t}] := \lim_{T\to 0}\frac{1}{T}\mathbb{E}_t[(W_{i,t+T} - W_{i,t})(W_{j,t+T} - W_{j,t})] = 1 - 6|i-j|(1-|i-j|).$$

and verifies property (ii).[50] Thus, W satisfies Assumption 1.

[50] Property (ii) of Assumption 1 was made to ensure $\int_0^1 (W_{i,t+T} - W_{i,t})di = 0$ almost-surely for any $T > 0$. Note that, with the definition of $W_{i,t}$ above, this integration can be verified directly.

C Extensions and Auxiliary Results

C.1 Financial Shocks and Risk Concentration

In the baseline model of Section 2, so-called risk concentration channel is absent, i.e., the wealth distribution evolves deterministically, which makes the entire economy evolve deterministically. Exactly as in Di Tella (2017), this conclusion arises due to the combination of agents' symmetric risk preferences and their ability to frictionlessly access markets for trading aggregate risk. However, as Di Tella (2017) goes on to show, idiosyncratic uncertainty shocks may break this neutrality result and lead financiers to take excess risk. In my model, excess financier risk is more likely to generated by diversification shocks than by uncertainty shocks (or LTV shocks).

To demonstrate this, I allow $(\Delta, \hat{\sigma}, \phi)$ to be truly stochastic aggregate shocks and show that shocks to Δ incentivize financiers to disproportionately incur aggregate risk, whereas shocks to $\hat{\sigma}$ and ϕ do not. Since financiers wealth tends to be more cyclical than households' empirically, this suggests that financial shocks like Δ are needed in model economies.

Consider the following setting. Take the economy from Section 2 and set $G_B = -\infty$ so that there is a single capital stock and single consumption good. Since there is only one type of insider, call them "households" and label their variables with the subscript "H". Endow financiers and households with symmetric Epstein-Zin preferences, as in (63). Suppose all agents have unitary EIS, i.e., $\varsigma = 1$.

Let $(\Delta, \hat{\sigma}, \phi)$ follow the following exogenous stationary Markov processes:

$$\begin{aligned} d\Delta_t &= \mu^{\Delta}(\Delta_t)dt + \sigma^{\Delta}(\Delta_t)dZ_t \\ d\hat{\sigma}_t &= \mu^{\hat{\sigma}}(\hat{\sigma}_t)dt + \sigma^{\hat{\sigma}}(\hat{\sigma}_t)dZ_t \\ d\phi_t &= \mu^{\phi}(\phi_t)dt + \sigma^{\phi}(\phi_t)dZ_t \end{aligned}$$

Note that shocks to $(\Delta_t, \hat{\sigma}_t, \phi_t)$ are locally perfectly correlated with aggregate TFP shocks, which is only for simplicity and does not affect the results of this section. In fact, if TFP shocks are shut down completely, everything here goes through. Assume $\sigma^{\Delta} \geq 0, \sigma^{\hat{\sigma}} \leq 0, \sigma^{\phi} \geq 0$ in accordance with conventional wisdom and empirical evidence.[51]

Because shocks to $(\Delta, \hat{\sigma}, \phi)$ are driven by the same aggregate shock as capital, the optimality conditions outlined in Appendix A.2 continue to hold here. The additional complication is that ξ_F and ξ_H, the processes measuring financier and household marginal utility of wealth, satisfy PDEs in $(\eta, \Delta, \hat{\sigma}, \phi)$ rather than ODEs in η. In equilibrium, we have the following idiosyncratic risk prices earned by financiers and households,

$$\hat{\pi}_{F,t} = \gamma \frac{(1-\Delta_t)\phi_t \hat{\sigma}_t}{\eta_t} \quad \text{and} \quad \hat{\pi}_{H,t} = \gamma \frac{(1-\phi_t)\hat{\sigma}_t}{1-\eta_t},$$

which are generalized from Section 2 because risk aversion $\gamma \neq 1$ and because $(\Delta_t, \hat{\sigma}_t, \phi_t)$ are non-constant.

[51] There is some controversy over $\sigma^{\phi} \geq 0$. In this model, external finance is done via equity contracts, and higher ϕ is a sign of a more efficient financial sector. From this perspective, the statement that ϕ is procyclical is intuitive. In the data, stock market equity-issuance is procyclical. Similarly, the loan-to-value ratios of the *marginal* borrower tend to be procyclical (e.g., Favilukis et al. (2017)). On the other hand, *aggregate* loan-to-value ratios in the housing market tend to be countercyclical (e.g., Davis and Van Nieuwerburgh (2015)). This discrepancy can be partially accounted for by the fact that borrowers typically issue debt contracts, whose values are mechanically less procyclical than levered equity. Still, note that if $\sigma^{\phi} \leq 0$, the results of this section on hedging demands for ϕ_t all flip signs, and shocks to ϕ_t can also be a source of risk concentration.

The idiosyncratic risk premia (risk price times risk quantity) earned on idiosyncratic risks are given by

$$\varpi_{F,t} = \gamma\Big(\frac{(1-\Delta_t)\phi_t\hat{\sigma}_t}{\eta_t}\Big)^2 \quad \text{and} \quad \varpi_{H,t} = \gamma\Big(\frac{(1-\phi_t)\hat{\sigma}_t}{1-\eta_t}\Big)^2.$$

At this point, we can understand the intuition for how Δ_t generates financial risk concentration. Notice that $\varpi_{H,t}$ is unaffected by Δ_t, whereas $\varpi_{F,t}$ is decreasing in Δ_t. If $\sigma_t^\Delta > 0$, negative shocks $dZ_t < 0$ will decrease Δ_t and lead to higher financier expected returns going forward, relative to households. Such an improvement of the relative investment opportunity set of financiers is a hedge and suggests that financiers' utility diffusion is smaller than households', i.e., $\sigma^\xi_{F,t} < \sigma^\xi_{H,t}$.

Shocks to Δ_t have aggregate implications through this mechanism. Indeed, clearing the aggregate risk market, we obtain

$$\pi_t = \gamma\sigma + (\gamma-1)\Big[(1-\eta_t)\sigma^\xi_{H,t} + \eta_t\sigma^\xi_{F,t}\Big].$$

Applying Itô's formula to η_t, we find that its local volatility is

$$\sigma^\eta_t = \eta_t(1-\eta_t)\frac{1-\gamma}{\gamma}\Big[\sigma^\xi_{F,t} - \sigma^\xi_{H,t}\Big].$$

If $\sigma^\xi_{F,t} < \sigma^\xi_{H,t}$, as conjectured above, then $\sigma^\eta_t > 0$ if and only if $\gamma > 1$. Hence, the hedging demands induced by financial shocks Δ_t create risk concentration in the sense that financiers buy Arrow claims on dZ_t from households. This stands in contrast to the situation in which Δ_t is non-stochastic as in Section 2. In that case, no shocks affect the relative investment opportunity sets of experts and financiers, so $\sigma^\eta_t \equiv 0$. This result echoes the results of Di Tella (2017): Δ_t impacts the uncertainty faced by financiers, just as a direct uncertainty shock does.

An exact opposite logic is true for shocks to ϕ_t. When $\sigma^\phi > 0$, negative shocks $dZ_t < 0$ will decrease ϕ_t and lead to lower financier expected returns going forward, relative to households. This exposure is the opposite of a hedge, and suggests $\sigma^\xi_{F,t} > \sigma^\xi_{H,t}$. If $\gamma > 1$ so that hedging demands are strong, then it is households who will hold concentrated aggregate risk positions and $\sigma^\eta < 0$.

The story is slightly different under uncertainty shocks (shocks to $\hat{\sigma}$). Uncertainty shocks would affect both ϖ_F and ϖ_H in the same direction, and the strength of this effect depends on the relative size of $(1-\Delta_t)\phi_t/\eta_t$ and $(1-\phi_t)/(1-\eta_t)$. Ultimately, this produces an ambiguous effect on financial risk concentration. In fact, we have the following neutrality result.

Proposition C.1 (Neutrality of Uncertainty Shocks)**.** *Consider an economy with only uncertainty shocks (i.e., $\mu^\Delta, \sigma^\Delta, \mu^\phi, \sigma^\phi \equiv 0$, while $\sigma^{\hat{\sigma}} < 0$). If at any point of time $\eta_t = \eta_\infty := \frac{\phi(1-\Delta)}{1-\phi\Delta}$, then there exists an equilibrium in which $\eta_t = \eta_\infty$ thereafter. If $\Delta < 1$ and $0 < \phi < 1$, in addition, then $\xi_F = \xi_H$ in this equilibrium.*

The key to Proposition C.1 is that the drift nets out financier and household risk compensation:

$$\mu^\eta = \gamma^{-1}\eta(1-\eta)[\hat{\pi}_F^2 - \hat{\pi}_H^2].$$

Since both $\hat{\pi}_F$ and $\hat{\pi}_H$ scale with $\hat{\sigma}$, uncertainty shocks do not affect the "steady state" of this system (defined by $\mu^\eta = 0$). At this steady-state $\eta = \eta_\infty$, idiosyncratic risk prices are equalized, $\hat{\pi}_F = \hat{\pi}_H$. Hence, shocks to $\hat{\sigma}$ affect both agents equally. Formally, one can verify that both agents' HJB equations and all equilibrium conditions are satisfied if $\xi_F = \xi_H$ at $\eta = \eta_\infty$. This leads to $\sigma^\xi_F(\eta_\infty) = \sigma^\xi_H(\eta_\infty)$ and thus a non-stochastic equilibrium with $\sigma^\eta(\eta_\infty) = 0$. This then verifies the conjecture that $\xi_F = \xi_H$, as η_t stays constant forever.[52]

[52]Furthermore, even starting with $\eta_t \neq \eta_\infty$, I conjecture that there are parameters such that $\eta_t \to \eta_\infty$ almost-surely.

The difference between Di Tella (2017) and this model is that $\Delta < 1$. When $\Delta = 1$, outsiders are unaffected by idiosyncratic risk; only insiders' investment opportunity sets are affected by uncertainty shocks. Mathematically, ξ_H contains more exposure to $\hat{\sigma}$ than ξ_F. In that case, uncertainty shocks provide insiders a hedge against bad states, and they will take additional aggregate risk ex-ante. But this result is fragile, as it is not generically true for any $\Delta < 1$. A similar discussion applies for the assumption $0 < \phi < 1$.

To provide an analytical result illustrating the induced hedging demands of (Δ, ϕ) shocks, we simplify the setting even further. Suppose at a known time τ, one of the following experiments will occur. We will consider these experiments one-by-one.

$$\text{(“}\Delta\text{ experiment”)}: \quad \Delta_\tau = \begin{cases} \Delta_+, & \text{with probability } 1/2 \\ \Delta_-, & \text{with probability } 1/2 \end{cases}$$

$$\text{(“}\phi\text{ experiment”)}: \quad \phi_\tau = \begin{cases} \phi_+, & \text{with probability } 1/2 \\ \phi_-, & \text{with probability } 1/2 \end{cases}$$

where $\Delta_+ > \Delta_-$ and $\phi_+ > \phi_-$. In each experiment, allow both financiers and households to frictionlessly trade Arrow claims on these shocks. Let the Arrow state prices be given by q_+ and q_-. After the shock, each variable will remain constant, i.e., $(\Delta_t, \phi_t) = (\Delta_\tau, \phi_\tau)$ for all $t \geq \tau$.

Thus, financiers and households each solve the following problem prior to the shock:

$$\max_{n_+, n_-} \frac{1}{2}\frac{(\xi_+ n_+)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\frac{(\xi_- n_-)^{1-\gamma}}{1-\gamma}$$
$$\text{subject to } q_+ n_+ + q_- n_- = n_{\tau-},$$

where n_+, n_- are net worths and ξ_+, ξ_- the marginal utility process in the $(+)$ and $(-)$ states at time τ. After the shock, the economy is assumed to be in a Markov equilibrium in the state variable η. Let $\xi_F(\eta_t; \Delta_t, \phi_t) := \xi_{F,t}$ and $\xi_H(\eta_t; \Delta_t, \phi_t) := \xi_{H,t}$ be the equilibrium marginal utility processes in that equilibrium for all $t \geq \tau$. We have the following results, the proofs of which are omitted because they follow closely Di Tella (2017).

Lemma C.2 (Relative Investment Opportunities)**.** *The function $\Xi(\eta; \Delta, \phi) := \xi_F(\eta; \Delta, \phi)/\xi_H(\eta; \Delta, \phi)$ is strictly decreasing in η, strictly decreasing in Δ, and strictly increasing in ϕ.*

Proposition C.3 (Risk Concentration)**.** *Suppose $\gamma > 1$. In the “Δ experiment,” we have $\eta_+ > \eta_{\tau-} > \eta_-$. In the “$\phi$ experiment,” we have $\eta_+ < \eta_{\tau-} < \eta_-$.*

In summary, by extending the logic of Proposition C.3, we expect that the financier wealth share η_t will be positively correlated with Δ_t shocks and negatively correlated with ϕ_t shocks. Procyclicality of Δ_t induces procyclicality of η_t – as in canonical models like Basak and Cuoco (1998), Brunnermeier and Sannikov (2014), and He and Krishnamurthy (2013) – but without assuming the typical exogenous trading restrictions. On the other hand, procyclicality of ϕ_t induces countercyclicality of η_t, opposite to these models. Finally, the cyclicality of $\hat{\sigma}_t$ is less likely to matter, due to the neutrality result of Proposition C.1, which is a key difference from Di Tella (2017).

Indeed, $\mu^\eta > 0$ for $\eta < \eta_\infty$ and $\mu^\eta < 0$ for $\eta > \eta_\infty$. Thus, η_∞ is an attracting point of the dynamical system. Whether the state variable hits the attracting point (and thus stays forever) depends on the relative speed at which μ^η and σ^η vanish when $\eta \to \eta_\infty$.

C.2 Endogenous Financial Innovation

To capture the idea that financial innovation can be an endogenous response to other shocks, I allow for the endogenous choice of diversification Δ. Diversification is chosen optimally by financiers to balance risk and exogenously specified costs. I assume that it is costly for a financier to participate in financial markets at all, and it is costlier to finance firms located far away from location i, which is embedded in a cost function. In particular, if intermediary i funds an arc of locations having length $\Delta_{i,t}$, she incurs the flow non-pecuniary cost $\frac{1}{2}\zeta(\Delta_{i,t})dt$. I make the following assumptions about the function ζ, borrowing from Gârleanu et al. (2015).

Assumption 5 (Diversification Costs)**.** *Assume the function $\zeta(\Delta)$ has the following properties:*

(i) $\zeta(\Delta) \geq 0$, $\zeta'(\Delta) \geq 0$, *and* $\zeta''(\Delta) > 0$ *for all* Δ.

(ii) $(1-\Delta)^3\zeta'(\Delta)$ *is increasing in* Δ *for all* Δ.

(iii) $\zeta(0) = \zeta'(0) = 0$.

First, whereas Lemma 2.2 shows the benefits of increasing Δ in the form of diversification, Assumption 5 describes the costs of diversification. Part (i) of the assumption says that the cost function is positive, and increasing and convex in distance. Part (ii) additionally ensures that the individual financier's problem is well-defined in that the first-order conditions of the problem are sufficient for optimality. Part (iii) ensures that some diversification is guaranteed.

Now, consider the model of Section 2 augmented with diversification costs. Financier's problem is augmented by these costs, and they now solve

$$\max \mathcal{U}^F_{i,t} - \frac{1}{2}\mathbb{E}_t \int_t^\infty e^{-\rho s}[\zeta_A(\Delta^A_{i,s}) + \zeta_B(\Delta^B_{i,s})]ds$$

subject to (9). Due to orthogonality of sectoral idiosyncratic shocks and additive separability of the diversification cost functions, let us focus on the financier investment in one of the sectors. Drop all sector subscripts/superscripts for the time being.

In a symmetric equilibrium, there is a location-invariant sectoral lending spread, which is unaffected by individual diversification choices. This means diversification is chosen to minimize the sum of portfolio variance and diversification costs ζ. The portfolio variance is given by $(\lambda\hat{\sigma}(1-\Delta))^2$, where λ is the volume of sector z lending per unit of net worth. This first-order condition says

$$\zeta'(\Delta) = (1-\Delta)(\lambda\hat{\sigma})^2.$$

Under Part (ii) of Assumption 5 implies that this condition is sufficient for optimality. Next, optimal risk-taking implies $\lambda = \frac{\hat{\pi}_F}{(1-\Delta)\hat{\sigma}}$, so

$$(1-\Delta)\zeta'(\Delta) = \hat{\pi}_F^2.$$

Finally, in equilibrium, $\hat{\pi}_F = \eta^{-1}\kappa\phi(1-\Delta)\hat{\sigma}$, so

$$\frac{\zeta'(\Delta)}{1-\Delta} = \left(\frac{\kappa\phi\hat{\sigma}}{\eta}\right)^2. \tag{105}$$

Assumption 5 implies this equation has a unique interior solution for Δ. Let this solution be denoted $\Delta(\alpha, \eta; \zeta)$ since (α, η) are the state variables in equilibrium.

We can do comparative statics on (105) for $\Delta(\alpha, \eta; \zeta)$. For example, we find that optimal Δ falls in response to a proportional reduction in the function $\zeta(\cdot)$ for a particular sector. A direct financial innovation

shock (akin to $\Delta \uparrow$ in the baseline model) can be achieved by a downward shock to the entire function $\zeta(\cdot)$. Lower diversification costs induce higher choices of diversification.

Moreover, the endogenous Δ now responds to other financial shocks as well. By re-incorporating the second sector and taking ratios, we have

$$\frac{\zeta_A'(\Delta_A)}{\zeta_B'(\Delta_B)}\frac{1-\Delta_B}{1-\Delta_A} = \Big(\frac{\kappa\phi_A\hat{\sigma}_A}{(1-\kappa)\phi_B\hat{\sigma}_B}\Big)^2. \tag{106}$$

The left-hand-side of (106) is increasing in Δ_A/Δ_B. Thus, we may think about how relative diversification incentives are affected by other shocks.

For example, consider $\phi_A \uparrow$ and $\hat{\sigma}_A \downarrow$, both of which generate reallocation towards sector A (i.e., $\kappa \uparrow$), as discussed in Section 4. An increase in ϕ_A, which increases sectoral borrowing from intermediaries, unambiguously increases relative diversification Δ_A/Δ_B. Conversely, a decrease in $\hat{\sigma}_A$, which increases all market participants' incentives to invest in the affected sector, may actually decrease financiers' relative diversification Δ_A/Δ_B. Endogenous diversification thus amplifies reallocation when $\phi_A \uparrow$ and dampens reallocation when $\hat{\sigma}_A \downarrow$. Intuitively, $\phi_A \uparrow$ is a credit demand shock, which induces a credit supply response, while $\hat{\sigma}_A \downarrow$ is a credit supply shock, which induces some endogenous retrenchment of credit supply as paying large diversification costs is no longer necessary. Lastly, any neutral credit supply shock, i.e., a shock that leaves κ relatively unaffected, cannot affect relative diversification motives.

C.3 Differentiated Goods

For analytical tractability, I have assumed that the consumption goods of sectors A and B are perfect substitutes. In this appendix, I allow the goods to be differentiated as a robustness exercise. In particular, I replace agents' utility functions (1) with

$$\mathcal{U}_t := \mathbb{E}_t\Big[\int_t^\infty \rho e^{-\rho(s-t)}\log(c_s)ds\Big],$$

where $c := a^{1-\beta}b^\beta$ is a Cobb-Douglas aggregate of the sector A good a and the sector B good b. Cobb-Douglas implies the two consumption goods have expenditure shares of $1-\beta$, β. I assume the composite good c is the numeraire. Let the relative prices of a and b be p_A and p_B. All other features of the model are unchanged.

With this modification, the equilibrium of Section 2 is modified as follows. First, the resource constraint from Definition 1 must replaced by three goods market clearing conditions:

$$\begin{aligned}
\int_0^1 G_A k_{i,t}^A di &= \int_0^1 [a_{i,t}^A + a_{i,t}^B + a_{i,t}^F]di \\
\int_0^1 G_B k_{i,t}^B di &= \int_0^1 [b_{i,t}^A + b_{i,t}^B + b_{i,t}^F]di \\
\int_0^1 (G_A k_{i,t}^A)^{1-\beta}(G_B k_{i,t}^B)^\beta di &= \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F]di + \int_0^1 [\iota_{i,t}^A k_{i,t}^A + \iota_{i,t}^B k_{i,t}^B]di.
\end{aligned}$$

The third condition aggregates output into the numeraire basket and splits this output into consumption and investment, which I assume is denominated in units of the numeraire.

Second, the equilibrium capital share κ and total capital growth ι are now determined via

$$\rho\Big[\frac{1-\beta}{\kappa}-\frac{\beta}{1-\kappa}\Big]-(\kappa\|\sigma_A\|^2-(1-\kappa)\|\sigma_B\|^2) \tag{107}$$
$$=\Big[(1-\phi_A)\hat{\pi}_A+\phi_A(1-\Delta_A)\hat{\pi}_{F\to A}\Big]\hat{\sigma}_A-\Big[(1-\phi_B)\hat{\pi}_B+\phi_B(1-\Delta_B)\hat{\pi}_{F\to B}\Big]\hat{\sigma}_B$$

and

$$\iota=(G_A\kappa)^{1-\beta}(G_B(1-\kappa))^{\beta}-\rho, \tag{108}$$

where $\hat{\pi}_A,\hat{\pi}_B,\hat{\pi}_{F\to A},\hat{\pi}_{F\to B}$ are given in (13)-(14). Equation (107) is a nonlinear equation, but it has a unique solution. Indeed, as $\kappa\to 0$ or $\kappa\to 1$, the left-hand-side converges to $+\infty$ and $-\infty$, respectively, whereas the right-hand-side stays bounded. Furthermore, the left-hand-side is strictly decreasing in κ, while the right-hand-side is strictly increasing in κ. Notice that, all else equal, Δ_A affects the equilibrium by reducing the right-hand-side of equation (107). Consequently, κ is increasing in Δ_A as before – the reallocation effect. The leverage effect survives because the formula for μ^η is unchanged.

Third, the goods prices are equilibrium objects. The price ratio is given by

$$\frac{p_{A,t}}{p_{B,t}}=\frac{1-\beta}{\beta}\frac{G_B}{G_A}\frac{1-\kappa_t}{\kappa_t}. \tag{109}$$

This "exchange rate" allows an international economics interpretation. One could interpret sector A as domestic producers and sector B as foreign producers, with funds intermediated by a single global financial sector. The presence of κ_t in $p_{A,t}/p_{B,t}$ implies exchange rates are determined by global capital flows, unlike a frictionless complete-markets economy. As κ_t is influenced by financial variables like Δ_A,Δ_B and intermediary wealth η_t, global financial shocks affect exchange-rate dynamics, similar to the intermediary-centric theoretical analysis of Gabaix and Maggiori (2015). A diversification boom in one country can thus have spillovers to the global economy, through leverage increases in the global financial system.

C.4 Investment Adjustment Costs

Suppose each sector now faces investment adjustment costs. For capital of sector $z\in\{A,B\}$ to grow by $\iota k dt$, an insider must pay $\Psi_z(\iota)kdt$ in investment costs. Standard q-theory holds, and optimal investment satisfies $\Psi_z'(\iota_{z,t})=q_{z,t}$, where q_z is the location-invariant unit price of installed capital in sector z. This capital price is presumed to follow a diffusion whose parameters will be determined in equilibrium:

$$dq_{z,t}=q_{z,t}[\mu^q_{z,t}dt+\sigma^q_{z,t}\cdot dZ_t].$$

I assume there is only one type of capital that can be freely traded across sectors, as in Brunnermeier and Sannikov (2015). What is really necessary in a broad bust is that capital is illiquid in the aggregate, rather than sector-by-sector. Under this assumption, $q_A=q_B\equiv q$, so it suffices to track

$$dq_t=q_t[\mu^q_t dt+\sigma^q_t\cdot dZ_t].$$

Thus, the return-on-capital from (5)-(6) are now given by

$$dR^z_{i,t}=\Big[\frac{G_z-\Psi_z(\iota_{z,t})}{q_t}+\iota_{z,t}+\mu^q_t+\sigma_z\cdot\sigma^q_t\Big]dt+\Big[\sigma_z+\sigma^q_t\Big]\cdot dZ_t+\hat{\sigma}_z dW^z_{i,t}.$$

Finally, the goods and bond market clearing conditions from Definition 1 are replaced by

$$\int_0^1 [G_A k_{i,t}^A + G_B k_{i,t}^B]di = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F + c_{i,t}^D]di + \int_0^1 [\Psi_A(\iota_{i,t}^A)k_{i,t}^A + \Psi_B(\iota_{i,t}^B)k_{i,t}^B]di$$
$$\int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F + n_{i,t}^D]di = q_t \int_0^1 [k_{i,t}^A + k_{i,t}^B]di.$$

I no longer assume any deadweight losses from distressed investor participation. Instead, participation cost χ is assumed to be a non-pecuniary flow cost.[53]

To obtain simple analytical solutions, I assume $\Psi_z(\iota) = \psi_z^{-1}[\exp(\psi_z \iota) - 1]$. With this functional form, it is possible to manipulate goods market clearing to solve for the symmetric equilibrium capital price:

$$q = \frac{\kappa(G_A + \psi_A^{-1}) + (1-\kappa)(G_B + \psi_B^{-1})}{\rho^* + \kappa\psi_A^{-1} + (1-\kappa)\psi_B^{-1}}, \tag{110}$$

where ρ^* is the aggregate consumption-wealth ratio. In the equilibrium of Proposition 3.1, the resource constraint is only ever used to obtain optimal investment. Now, it is used to obtain the equilibrium capital price, and optimal investment is given by $\iota_A = \psi_A^{-1}\log(q)$ and $\iota_B = \psi_B^{-1}\log(q)$.

The rest of the equilibrium is determined as follows. The pricing equations for in each sector are given by

$$\begin{aligned}\frac{G_A - \psi_A^{-1}(q-1)}{q} + \psi_A^{-1}\log(q) + \mu^q + \sigma_A \cdot \sigma^q - r \geq{}& (1-\phi_A)[\sigma_A + \sigma^q]\cdot\pi_A + \phi_A[\sigma_A + \sigma^q]\cdot\pi \\ &+ (1-\phi_A)\hat{\sigma}_A\hat{\pi}_A + \phi_A(s_A - [\sigma_A + \sigma^q]\cdot\pi)\\
\frac{G_B - \psi_B^{-1}(q-1)}{q} + \psi_B^{-1}\log(q) + \mu^q + \sigma_B \cdot \sigma^q - r \geq{}& (1-\phi_B)[\sigma_B + \sigma^q]\cdot\pi_B + \phi_B[\sigma_B + \sigma^q]\cdot\pi \\ &+ (1-\phi_B)\hat{\sigma}_B\hat{\pi}_B + \phi_B(s_B - [\sigma_B + \sigma^q]\cdot\pi),\end{aligned}$$

where π_A and π_B are insiders' shadow aggregate risk prices, π is the traded aggregate risk price, and $\hat{\pi}_A, \hat{\pi}_B, \hat{\pi}_{F\to A}, \hat{\pi}_{F\to B}$ are idiosyncratic risk prices defined in Proposition 3.1. The formula for equilibrium spreads is the same as Proposition 3.1 with σ_z replaced by $\sigma_z + \sigma^q$:

$$\begin{aligned}s_z - [\sigma_z + \sigma^q]\cdot\pi = {}& x\Big[(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z} + \zeta - \big(\zeta - \chi - \frac{x}{1-x}(1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big] \\ &+ (1-x)\Big[\chi - \big(\chi - \zeta - (1-\Delta_z)\hat{\sigma}_z\hat{\pi}_{F\to z}\big)^+\Big], \quad z \in \{A, B\},\end{aligned} \tag{111}$$

Take the difference between the sectoral pricing equations (assuming equality) to obtain

$$\begin{aligned}H := {}& \frac{G_A - G_B - (\psi_A^{-1} - \psi_B^{-1})(q-1)}{q} + (\psi_A^{-1} - \psi_B^{-1})\log(q) + (\sigma_A - \sigma_B)\cdot\sigma^q \\ &- (1-\phi_A)[\sigma_A + \sigma^q]\cdot\pi_A - \phi_A[\sigma_A + \sigma^q]\cdot\pi - (1-\phi_A)\hat{\sigma}_A\hat{\pi}_A - \phi_A(s_A - [\sigma_A + \sigma^q]\cdot\pi) \\ &+ (1-\phi_B)[\sigma_B + \sigma^q]\cdot\pi_B + \phi_B[\sigma_B + \sigma^q]\cdot\pi + (1-\phi_B)\hat{\sigma}_B\hat{\pi}_B + \phi_B(s_B - [\sigma_B + \sigma^q]\cdot\pi).\end{aligned}$$

Since q is determined by (110), μ^q and σ^q can be obtained by Itô's formula. Therefore, H is an explicit function of the state variables (α, η, x), conditional on (κ, ζ). As in Proposition 3.1, (κ, ζ) are determined by

[53] To model this, we must assume distressed investors' period utility function is augmented by $-\chi\Xi_{i,t}(\lambda_{D,i,t}^A + \lambda_{D,i,t}^B)dt$, where $\Xi_{i,t}$ is an appropriately-chosen process. For example, with Epstein-Zin preferences as in Appendix A.2, we must choose $\Xi_{i,t} := (\xi_t n_{i,t}^D)^{1-\gamma}$ in order for χ to be perceived as a scale-invariant reduction in investment returns. With non-pecuniary costs of this form, one can verify that their portfolio choices are exactly the same as in Appendix A.2.

the nonlinear system

$$0 = \min\left\{\zeta,\, \bar{\lambda} - \lambda_F^A - \lambda_F^B\right\}$$
$$0 = \min\left\{1-\kappa,\, H^+\right\} - \min\left\{\kappa,\, (-H)^+\right\},$$

where H is defined above.

C.5 Busts as Flight-to-Safety Episodes

Here, we consider an alternative model which also generates busts due to financiers' leverage constraint. Consider the economy from Section 3 without distressed investors (i.e., set $\chi = +\infty$) and without the overlapping generations structure (i.e., set $\delta = 0$). Introduce a new production technology as follows.

With $k \geq 0$ units of capital, this technology produces $\underline{r}k$, with $\underline{r} < \min(G_A, G_B)$. Despite being less productive, this technology is safer, in the sense that capital is riskless while being used as an input in this technology. There is a rental market for this capital, which must have rental rate $\underline{r}$, because of the linear production technology and absence of capital-quality shocks.

In this economy, the goods and bond market clearing conditions are modified to read

$$\int_0^1 [G_A k_{i,t}^A + G_B k_{i,t}^B + \underline{r}\,\underline{K}_t]di = \int_0^1 [c_{i,t}^A + c_{i,t}^B + c_{i,t}^F]di + \frac{1}{dt}\int_0^1 [dI_{i,t}^A + dI_{i,t}^B + d\underline{I}_{i,t}]di$$
$$\int_0^1 [n_{i,t}^A + n_{i,t}^B + n_{i,t}^F]di = \int_0^1 [k_{i,t}^A + k_{i,t}^B + \underline{K}_t]di,$$

where $\underline{K}_t$ and $d\underline{I}_t$ are the capital stock and investment flow into the riskless sector. As before, I will study a symmetric equilibrium, with state variables $K_t := \int_0^1 [k_{i,t}^A + k_{i,t}^B + \underline{K}_t]di$, $\eta_t := (\int_0^1 n_{i,t}^F di)/K_t$, and $\alpha_t := (\int_0^1 n_{i,t}^A di)/(\int_0^1 [n_{i,t}^A + n_{i,t}^B]di)$.

Using the definition of the aggregate investment rate ι_t and optimality conditions for consumption, we may re-write the goods market clearing condition as

$$\omega[\kappa G_A + (1-\kappa)G_B] + (1-\omega)\underline{r} = (1-\eta)\rho + \eta\rho_F + \iota, \tag{112}$$

after scaling by K_t. In (112), ω is the capital share in either of the risky technologies A or B, whereas κ is the share, among the risky capital, in sector A. It is clear that, holding fixed η and κ, a decline in ω must reduce economic growth ι.

Finally, I allow any agent to access this technology, which implies $r_t \geq \underline{r}$ by absence of arbitrage. This implies the complementary-slackness condition

$$r_t \geq \underline{r}, \quad \omega_t \leq 1, \quad \text{and} \quad (1-\omega_t)(r_t - \underline{r}) = 0. \tag{113}$$

Because safe capital allocation implies lower production and growth, I will refer to times with $\omega_t < 1$ as periods of *misallocation*.

By repeating the arguments of Propositions 3.1 and A.3, we may characterize the equilibrium. First, we find that equation (24) for ζ still holds. However, we must modify equation (27) for κ to account for the fact that risk prices are now scaled by the risky capital share ω. Indeed, insiders' and financiers' idiosyncratic risk

prices are now given by

$$\omega\hat{\pi}_{F\to A}, \quad \omega\hat{\pi}_{F\to B}, \quad \omega\hat{\pi}_A, \quad \text{and} \quad \omega\hat{\pi}_B,$$

where $\hat{\pi}_{F\to A}, \hat{\pi}_{F\to B}, \hat{\pi}_A, \hat{\pi}_B$ are defined in (13) and (14). Similarly, aggregate risk prices are now given by

$$\omega\pi, \quad \omega\pi_A, \quad \text{and} \quad \omega\pi_B,$$

where π, π_A, π_B are defined in the statement of Proposition 3.1. Equilibrium spreads are now given by

$$\begin{aligned} s_A - \omega\sigma_A \cdot \pi &= \zeta + \omega(1-\Delta_A)\hat{\sigma}_A\hat{\pi}_{F\to A} \\ s_B - \omega\sigma_B \cdot \pi &= \zeta + \omega(1-\Delta_B)\hat{\sigma}_B\hat{\pi}_{F\to B}. \end{aligned}$$

Using those definitions, we have the following equation for κ:

$$\begin{aligned} 0 &= \min\left\{1-\kappa,\, H^+\right\} - \min\left\{\kappa,\, (-H)^+\right\} \\ H &:= G_A - G_B - \phi_A s_A + \phi_B s_B - \omega(1-\phi_A)[\sigma_A \cdot \pi_A + \hat{\sigma}_A\hat{\pi}_A] + \omega(1-\phi_B)[\sigma_B \cdot \pi_B + \hat{\sigma}_B\hat{\pi}_B]. \end{aligned} \tag{114}$$

The equilibrium can be characterized by taking κ as given and solving for (ω, ζ, r). The result is in the following lemma.

Lemma C.4. *Equilibrium in the model with a positive-net-supply safe technology requires*

$$\omega = \min(1,\, \bar{\omega},\, \omega^*(0)) \tag{115}$$

$$\zeta = \mathbf{1}_{\{\bar{\omega}\le 1\}} \max(0,\, \zeta^*) \tag{116}$$

$$r = \mathbf{1}_{\{\omega<1\}}\underline{r} + \mathbf{1}_{\{\omega=1\}} r^*(\zeta), \tag{117}$$

where

$$\bar{\omega} := (\kappa\phi_A + (1-\kappa)\phi_B)^{-1}\eta\bar{\lambda} \tag{118}$$

$$\omega^*(\zeta) := \frac{\kappa G_A + (1-\kappa)G_B - \underline{r} - \zeta(\kappa\phi_A + (1-\kappa)\phi_B)}{\eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) + (1-\eta)[\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B)]} \tag{119}$$

$$\begin{aligned} \zeta^* := \frac{1}{\kappa\phi_A + (1-\kappa)\phi_B}\Big[&\kappa G_A + (1-\kappa)G_B - \underline{r} - \bar{\omega}\eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) \\ &- \bar{\omega}(1-\eta)(\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B))\Big] \end{aligned} \tag{120}$$

$$\begin{aligned} r^*(\zeta) := \kappa G_A + (1-\kappa)G_B - \zeta(\kappa\phi_A + (1-\kappa)\phi_B) \\ - \eta(\|\pi\|^2 + \hat{\pi}^2_{F\to A} + \hat{\pi}^2_{F\to B}) - (1-\eta)[\alpha(\|\pi_A\|^2 + \hat{\pi}^2_A) + (1-\alpha)(\|\pi_B\|^2 + \hat{\pi}^2_B)]. \end{aligned} \tag{121}$$

Proof. When the leverage constraint in (24) binds, we may substitute financiers' optimal portfolios, along with equilibrium spreads, to get $\omega = \bar{\omega}$ from (118). At the same time, we may sum insiders' pricing conditions, the generalizations of (81) and (82) which account for ω, each weighted by κ and $1-\kappa$ respectively, to construct an equation $F(\omega, \zeta, r) = 0$ that holds in equilibrium. When $\omega = 1$, we have $F(1, \zeta, r) = 0$. This defines $r^*(\zeta)$ in (121). When $\omega < 1$, we have $r = \underline{r}$ by complementary-slackness condition (113), and so $F(\omega, \zeta, \underline{r}) = 0$. This defines $\omega^*(\zeta)$ in (119). Note that, by their definitions, $\omega^*(\zeta) \le 1$ if and only if $r^*(\zeta) \le \underline{r}$. Finally, define ζ^* by (120), which is the solution to $\bar{\omega} = \omega^*(\zeta)$.

Armed with these arguments, consider the following mutually exclusive and completely exhaustive cases:

- Suppose $\bar{\omega} > 1$. Then, $\eta^{-1}(\kappa\phi_A + (1-\kappa)\phi_B) < \bar{\lambda}$, so (24) implies $\zeta = 0$. Thus, equation (116) holds in this case. Consider the following sub-cases.
  - Suppose $\omega^*(0) \geq 1$. This implies $\omega = 1$ in which case $r = r^*(0)$. Thus, (115) and (117) hold.
  - Suppose $\omega^*(0) < 1$. This implies $\omega = \omega^*(0)$ in which case $r = \underline{r}$. Thus, (115) and (117) hold.
- Suppose $\bar{\omega} \leq 1$. Consider the following sub-cases.
  - Suppose $\omega^*(0) \geq \bar{\omega}$. Then, $\zeta^* \geq 0$. Hence, we may set $\omega = \bar{\omega}$, $\zeta = \zeta^*$, and $r = \underline{r}$ to satisfy (115), (116), and (117). To see that this is the unique option, notice that either $\omega = \omega^*(0)$ or $\omega = 1$ imply $\omega\eta^{-1}(\kappa\phi_A + (1-\kappa)\phi_B) \geq \bar{\lambda}$, which is a violation of (24) except in the trivial case $\omega = \bar{\omega}$.
  - Suppose $\bar{\omega} > \omega^*(0)$. Then, $\zeta^* < 0$. Hence, we are required to set $\omega = \omega^*(0)$, $\zeta = 0$, and $r = \underline{r}$ in order to satisfy (115), (116), and (117).

This completes the proof, since (115), (116), and (117) hold in all cases. □

To solve for equilibrium κ, we solve equation (114), using the results of Lemma C.4. This is a nonlinear equation, which must in general be solved numerically. In the case where sectors A and B are exactly symmetrical (i.e., $G_A = G_B$, $\phi_A = \phi_B$, $\hat{\sigma}_A = \hat{\sigma}_B$, $\|\sigma_A\| = \|\sigma_B\|$, and $\Delta_A = \Delta_B$), the equilibrium simplifies as the solution must be $\kappa = 1/2$ from (114). In this simple case, we can clearly illustrate the flight-to-safety episode induced by the presence of the leverage constraint.

Indeed, we have

$$\bar{\omega} < 1 \Leftrightarrow \eta < \eta_1^* := \phi/\bar{\lambda}$$

and

$$\bar{\omega} < \omega^*(0) \Leftrightarrow \eta < \eta_2^* := \frac{(G-\underline{r})\phi/\bar{\lambda} - \frac{1}{2}\phi^2(\|\sigma\|^2 + (1-\Delta)^2\hat{\sigma}^2)}{(G-\underline{r})\phi/\bar{\lambda} - \frac{1}{2}\phi^2(\|\sigma\|^2 + (1-\Delta)^2\hat{\sigma}^2) + (1-\phi)^2(\|\sigma\|^2 + \hat{\sigma}^2)}.$$

Thus, $\omega = \bar{\omega} < 1$ if and only if[54]

$$\eta < \eta_1^* \wedge \eta_2^*.$$

Thus, the "leverage effect" we have described in the main text can lead to real effects through misallocation. If diversification Δ increases enough, such that η_t drifts downwards, eventually financiers' leverage constraints will bind. Binding constraints imply misallocation in the sense that $\bar{\omega} < 1$. This is the notion of "endogenous bust" discussed in the main text. Furthermore, a subsequent negative aggregate shock lowers η_t even more, which reduces ω_t one-for-one, evidently from equation (118). This is the leverage-induced "instability" discussed in the main text. In summary, analogously to the main text, a model with the possibility of flight-to-safety can generate diversification-induced cycles which are both endogenous and unstable.

[54] Note that if

$$(\phi/\bar{\lambda})(1-\phi/\bar{\lambda})(G-\underline{r}) > (\phi/\bar{\lambda})(1-\phi)^2(\|\sigma\|^2 + \hat{\sigma}^2) + (1-\phi/\bar{\lambda})\frac{\phi^2}{2}(\|\sigma\|^2 + (1-\Delta)^2\hat{\sigma}^2),$$

then $\eta_1^* < \eta_2^*$, so that a binding leverage constraint and and misallocation are equivalent, i.e., they occur at exactly the same times. This result is analogous to part (ii) of Proposition 3.5.

D Empirical Analysis

D.1 Historical Applications of Diversification Improvements

In the introduction, I have discussed the plausibility of a diversification-fueled boom in US housing markets during the 1990s-2000s. Mortgage securitization occurred at least four other times in US history: in the 1850s, 1880s, 1920s, and 1980s. In the 1850s, a market developed for securitization of railroad-adjacent farm loans, coinciding with a US railroad boom and preceding the broader Panic of 1857.[55] In the 1880s, US agriculture boomed as farm mortgages were increasingly securitized, preceding the Panic of 1893.[56] In the 1920s, prior to the Great Depression, residential and commercial real estate loans securitization boomed, coinciding with a construction boom.[57]

In the 1980s, financial sector diversification increased for two separate reasons. First, a market for prime-rated mortgage securitization emerged as government-sponsored enterprises (GSEs) Fannie Mae and Freddie Mac increased their participation in residential mortgage markets.[58] Second, state-level deregulations of the commercial banking sector integrated many local lending markets, and this may have disproportionately applied to household finance.[59] The 1980s boom ended in a broad bust, the so-called savings and loan crisis. My model could apply to these four episodes, insofar as diversification waves generate better risk-sharing for financiers in these markets.

D.2 Qualitative Support: Why the Model Applies to the US Housing Cycle

In addition to the *reallocation* and *leverage* patterns documented in figure 1, here I provide some more qualitative support for the mechanism of the model. First, the model requires that the increase in securitization actually improves diversification of mortgage loans. This is not necessarily true a priori: one possibility is that securitization of mortgage loans increases simply because the volume of mortgage lending increases. Figure 21 rejects this by showing that RMBS increase dramatically as a share of total household credit in the US. Moreover, non-agency MBS rise as a share of all MBS. Private label securitizations may be particularly important for diversification, because prior to the securitization boom, the types of loans in these pools were those most likely to be held on banks' balance sheets until maturity.

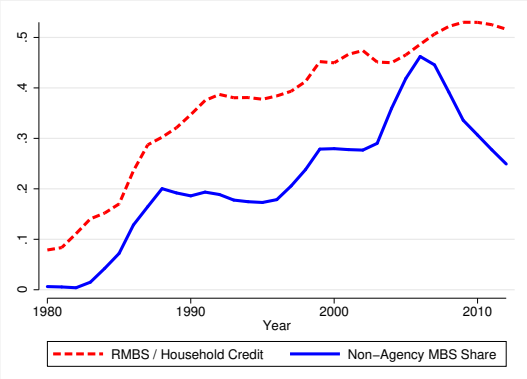

Figure 21: Securitization of household credit. "RMBS / Household Credit" sums both agency and non-agency RMBS and divides by the total household credit outstanding. "Non-Agency MBS Share" divides non-agency MBS by total MBS outstanding. Source: SIFMA and Flow of Funds.

[55] See Riddiough and Thompson (2012) and Calomiris and Schweikart (1991).
[56] See Eichengreen (1984), Snowden (1995), and Snowden (2007).
[57] See Goetzmann and Newman (2010), White (2009), Snowden (2010), and Eichengreen and Mitchener (2003).
[58] See Fieldhouse, Mertens and Ravn (2018).
[59] See Mian, Sufi and Verner (2017b) for cross-sectional evidence that these deregulations caused a credit boom and made the subsequent bust larger.

Second, it is crucial for my results that diversification in the housing market increases more than diversification in the corporate credit market. This turns out to be true, if we measure diversification by securities, which are likely to be broadly held. Figure 22 shows that mortgage securities outstanding were equal to corporate securities outstanding in 1990, but nearly double by 2007.

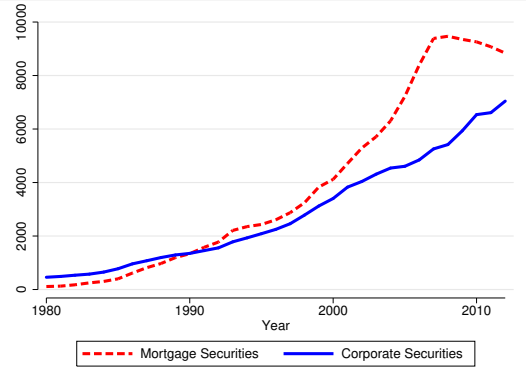

Figure 22: Mortgage versus Corporate Securities. "Mortgage Securities" are traded securities where the underlying assets are mortgages. "Corporate Securities" sums corporate bonds and any securitized bank loans. Source: SIFMA.

Third, my model assumes that the financial sector will adapt to an environment with better mortgage diversification by taking more housing-related risks onto their balance sheets. Figure 23 shows that commercial banks do indeed hold more housing-related assets on their balance sheets through the housing boom. Notice this series qualitatively mimics the household credit share from figure 1.

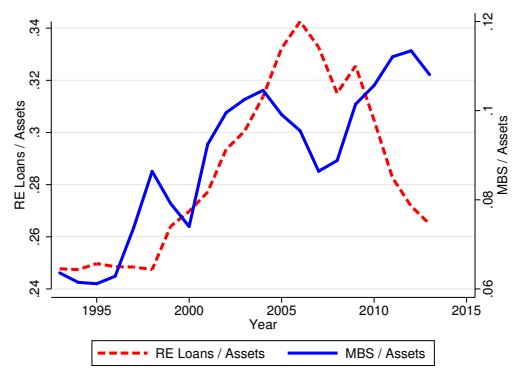

Figure 23: Commercial bank risk-taking in housing markets. "RE Loans / Assets" refers to real estate loans held on bank balance sheets, relative by assets. "MBS / Assets" are mortgage-backed securities held, relative to assets. Source: Call Reports.

Similarly, figure 24 shows that price-to-cash-flow ratios in capital and housing markets do not move in lockstep, suggestive of some sectoral asymmetry in this boom period.

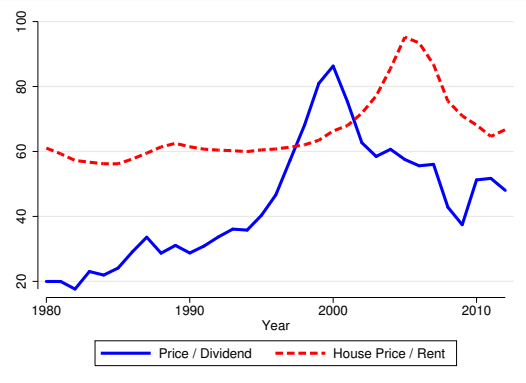

Figure 24: The price-dividend ratio on the S&P 500 and a measure of house prices relative to the rental rate on housing services. The house price-rent ratio is obtained from `http://datatoolkits.lincolninst.edu/subcenters/land-values`. The plotted ratio is scaled by 3.

Finally, a key reason financial sector capitalization deteriorates in my model is through declining financier profitability. As diversification improves in the model, financiers are willing to accept lower risk premia on mortgages. Figure 25 shows that commercial banks' profitability declined marginally between the boom years 2000-2007.

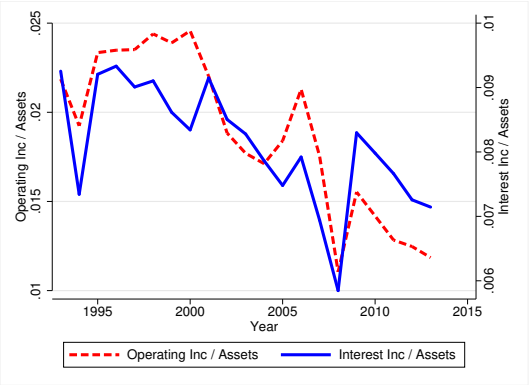

Figure 25: Commercial bank profitability. "Operating Inc / Assets" is operating income, relative to assets. "Interest Inc / Assets" is income from interest payments, relative to assets. Source: Call Reports.

D.3 Quantifying Mortgage Diversification

In this appendix, I describe more specifically the methodology to compute the diversification index of Section 5.1. Start by defining an aggregate mortgage return during month k of year t:

$$\overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} := \sum_{\ell} \omega_{\ell,t} R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

where $\omega_{\ell,t} := \frac{s_{\ell,t}+m_{\ell,t}}{\sum_{\ell'} s_{\ell',t}+m_{\ell',t}}$ are origination weights:

$$\begin{aligned} m_{\ell,t} &:= \text{ portfolio mortgages originated to location } \ell \text{ in year } t \\ s_{\ell,t} &:= \text{ securitized mortgages originated to location } \ell \text{ in year } t. \end{aligned}$$

The location-specific mortgage return $R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}$ is proxied by the housing return in location ℓ and month k of year t, taken from CoreLogic. This is the return building block for all other returns. The aggregate return $\overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$ allows me to extract the idiosyncratic components of all other returns.

In an analogous fashion, define the mortgage return for intermediary i:

$$R^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} := \sum_{\ell} \omega^{(i)}_{m,\ell,t} R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} + \omega^{(i)}_{s,\ell,t} \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$$

where

$$\omega^{(i)}_{m,\ell,t} := \frac{m^{(i)}_{\ell,t}}{\sum_{\ell'} s^{(i)}_{\ell',t} + m^{(i)}_{\ell',t}} \quad \text{and} \quad \omega^{(i)}_{s,\ell,t} := \frac{s^{(i)}_{\ell,t}}{\sum_{\ell'} s^{(i)}_{\ell',t} + m^{(i)}_{\ell',t}}$$

$$\begin{aligned} m^{(i)}_{\ell,t} &:= \text{ portfolio mortgages originated by lender } i \text{ to location } \ell \text{ in year } t \\ s^{(i)}_{\ell,t} &:= \text{ sold mortgages originated by lender } i \text{ to location } \ell \text{ in year } t. \end{aligned}$$

Note that any mortgages originated by intermediary i which are then sold off within the same year are captured by $s^{(i)}_{\ell,t}$. I make the assumption that these sales return the aggregate return $\overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}$, which

is subject to no idiosyncratic risk. Loans not sold are captured by $m_{\ell,t}^{(i)}$. These loans are assumed to get the location-specific return $R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}$.

Next, I define "idiosyncratic returns" by subtracting the aggregate return:

$$\begin{aligned}\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} &:= R_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}} - \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\\ \mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} &:= R^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}} - \overline{R}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\end{aligned}$$

The (monthly) idiosyncratic variances in year t are then given by

$$\begin{aligned}\mathcal{V}^2_{\ell,t} &:= \frac{1}{12}\sum_{k=1}^{12}\left(\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2 - \left(\frac{1}{12}\sum_{k=1}^{12}\mathcal{R}_{\ell,t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2\\ \mathcal{V}^2_{i,t} &:= \frac{1}{12}\sum_{k=1}^{12}\left(\mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2 - \left(\frac{1}{12}\sum_{k=1}^{12}\mathcal{R}^{(i)}_{t+\frac{k-1}{12}\to t+\frac{k}{12}}\right)^2\end{aligned}$$

In this computation, I am using the fact that the returns are computed monthly, while the originations and securitizations data are only available at a yearly frequency.

I average over locations and intermediaries to get the volatilities that I want:

$$\begin{aligned}\hat{\sigma}_t &:= \sum_{\ell}\omega_{\ell,t}\mathcal{V}_{\ell,t}\\ \hat{\sigma}_{\Delta,t} &:= \sum_{i}\omega_{i,t}\mathcal{V}_{i,t}\end{aligned}$$

where

$$\omega_{i,t} := \sum_{\ell}\frac{m^{(i)}_{\ell,t}+s^{(i)}_{\ell,t}}{\sum_{\ell',j}m^{(j)}_{\ell',t}+s^{(j)}_{\ell',t}}.$$

Note that we necessarily have $\hat{\sigma}_{\Delta,t}\leq\hat{\sigma}_t$, because correlations between the loans in lender's portfolios are less than 1, while loan-level volatilities are proxied by location-specific volatilities.

Finally, in the symmetric equilibrium, the following equation relates financiers' housing risk $\hat{\sigma}_\Delta$ to the fundamental housing risk $\hat{\sigma}$ and the level of diversification Δ:

$$(1-\Delta)\hat{\sigma} = \hat{\sigma}_\Delta = \text{ idio volatility of unlevered mortgage portfolio.}$$

Thus, I define

$$\Delta_t := 1 - \hat{\sigma}_{\Delta,t}/\hat{\sigma}_t$$

by inverting this relation. The units of Δ_t are the fraction of fundamental housing risk that are eliminated from lender's portfolios, either through loan sales and securitizations, or through geographic diversification.

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