From Lombard Street to Wall Street: Systemic Risk, Rescues, and Stability in Financial Networks*

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Abstract

Banks' obligations to each other involve counterparty risks. In case of a failure, the losses of counterparties of the failing bank are exacerbated by additional bankruptcy costs. By also taking the contagion risk into account, banks' potential losses incentivize them to rescue each other whenever rescues are less costly than absorbing the losses. Endogenously arising rescues reverse the standard intuitions from the financial contagion literature: A system-wide contagion risk does not necessarily imply financial instability and, surprisingly, leads to greater stability in certain networks where banks more than undo the contagious failures and take actions against any potential failure. In a framework where capital transfers between banks are more efficient than government bailouts, I characterize welfare-maximizing networks and show that they are connected through *i)* intermediate levels of interbank liabilities per bank, and *ii)* no clustering of interbank exposures among any subset of banks. Consequently, financial stability is determined by the potential bankruptcy losses internalized by banks and the loss absorption capacity of the system (i.e., banks' aggregate capital). The results provide additional insights into the historical debate on bank rescues and help us better understand the implications of current interbank regulations. The findings also offer plausible explanations for the selective rescues in the 2007-2009 period.

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1 Introduction

Financial institutions are interconnected through various types of contracts and common asset exposures (e.g., overnight or long-term loans, repo transactions, syndicate loans, swaps and other derivative contracts, collateralized debt obligations (CDOs), and asset-backed securities). In the first half of 2009, gross credit exposures of dealer banks in the over-the-counter (OTC) market were more than 3.7 trillion dollars and the total volume of transactions in the OTC market was about 600 trillion dollars (BIS [11]). As of the first quarter of 2019, the value of total claims of financial institutions among each other has exceeded 9.1 trillion dollars (BIS [12]). Interconnectedness across different financial institutions, which I call *banks* henceforth, serves as an insurance mechanism via liquidity provision or risk sharing, yet brings its own risks. The underlying causes and consequences of various types of risks associated with interconnectedness have been extensively studied in the literature.[1]

Even though the systemic consequences of interconnectedness have attracted considerable attention in the literature, the studies on how banks take actions against such systemic consequences are limited.[2] In this paper, I develop a framework for analyzing financial stability in the presence of counterparty risks, while considering that banks can form rescue mergers or rescue consortia in order to prevent costly bankruptcies and their systemic effects.[3]

The main insight behind banks' contributions in the rescue of distressed banks is as follows. The failure of a bank, if not prevented, results in the default of that bank's obligations to other banks, where additional bankruptcy (or liquidation) costs exacerbate the losses of its counterparties and might even lead to further losses in the rest of the system via domino effects. Such risks incentivize banks to rescue each other if the rescue costs for each participant is less than the potential losses that would otherwise occur. Moreover, rescues are not necessarily designed to prevent contagion. As long as it is profitable to do so, a rescue is designed to save a bank that would not cause any domino effects but would pose sufficiently high losses to the system otherwise. Consequently, in the presence of counterparty risks, *bank*

[1] Following the early contributions by Allen and Gale [8], Freixas, Parigi, and Rochet [36], Rochet and Tirole [60], and Kiyotaki and Moore [53]; there is a growing literature in financial networks including Acemoglu, Ozdaglar, and Tahbaz-Salehi ([1] and [2]), Allen and Babus [5], Allen, Babus, and Carletti [7], Cabrales, Gottardi and Vega-Redondo [17], Chang and Zhang [20], Elliott, Golub, and Jackson [31], Elliott, Hazell, and Georg [30], Erol [32], Erol and Vohra [33], Ibragimov, Jaffee and Walden [47], Farboodi [35], Gai and Kapada [37], Amini and Minca [9], Glasserman and Young [43], Galeotti, Ghiglino and Goyal [40], Battiston et al. [14], Cabrales, Gale, and Gottardi [16], Gofman [44], Corbae and Gofman [19], Jackson and Pernoud [49], Minca and Sulem [56], Nier et al. [57], Wang [65]. Demange [23], Diebold and Yilmaz [25], Gai, Haldane, and Kapadia [38], Cohen-Cole, Patacchini and Zenou [18], Eisenberg and Noe [29], Upper and Worms [64], Duffie and Wang [26], Duffie and Zhu [27]. Other types of difficulties can emerge in distressed times such as freezes in interbank lending, repo market or OTC market. See Di Maggio and Tahbaz-Salehi [28], Acharya, Gale and Yorulmazer [4], and Gorton and Metrick [45] for detailed discussions on such market freezes.

[2] Leitner [54], Rogers and Veraart [59], and Bernard, Capponi, and Stiglitz [15] are the related studies on bank rescues and contagion.

[3] Hoggarth, Reidhill and Sinclair [46] and White and Yorulmazer [66] discuss bank resolution concepts in detail.

rescues are similar to *private provision of public goods*[4] in the sense that all banks in the system, including non-counterparties of a distressed bank, weakly benefit from any avoided bankruptcy; however, rescuing a distressed bank requires transfers that are costly for the contributing banks. As a result, each bank's incentives depend on the potential system-wide consequences of interbank exposures, whereas capabilities of banks depend on the available capital in the system that can be used for such capital injections.

Rescue of distressed banks by other banks have repeatedly occurred throughout the history of the financial system with or without government assistance. Bank of England's coordination of Barings' rescue in 1890 is an example of a successful case. Similarly, the resolution of the Alsatian crisis in 1828 and the Hamburg crisis of 1857 illustrate the role of the private sector in handling distress and restoring confidence in the economy.[5] In the second half of the 19^{th} century and early 20^{th} century, New York Clearing House and other clearinghouse associations, which are private institutional frameworks, were resolving distress and restoring confidence in the US financial system, before the foundation of the Federal Reserve System in 1913.[6]

Recently, the resolution of Long-Term Capital Management (LTCM) in 1998 is an example of a multi-bank rescue consortium. The 14-member consortium injected around 3.6 billion dollars to save LTCM, with no government-assistance.[7]

On the other hand, the story in the 2007-2009 financial crisis was quite different than the LTCM case. In particular, rescue mergers were very effective in maintaining financial stability in the 2007-2009 period, but in the meantime some large banks were not rescued. Geithner [42] explains the rescue attempts in the Lehman Brothers case as follows:

[4]For surveys on games on networks, see Jackson [48] and Jackson and Zenou [50]. In a recent paper, Galeotti, Golub, and Goyal [41] study the design of optimal interventions in network games, where individuals' incentives to act are affected by their network neighbors' actions.

[5]See Kindleberger and Aliber [52] for further discussion on these two cases as well as the history of financial crises.

[6]In a recent work, Anderson, Erol, and Ordoñez [6] study the implications of the founding of the Federal Reserve System. Besides, James [51] explain the effectiveness of rescue mergers in the early 19^{th} as follows:

> "In some communities financial reconstruction was attempted by arrangements for a strong bank to merge with a weakened bank or, if several weakened banks were involved, by establishing a new institution with additional capital to take over the liabilities of the failing banks, the stockholders of which took a loss." (James, 1938)

[7]In the Report of the Presidents' Working Group on Financial Markets [61], it is explained as follows:

> "The firms in the consortium saw that their losses could be serious, with potential losses to some firms amounting to $300 million to $500 million each...The self-interest of these firms was to find an alternative resolution that cost less than they could expect to lose in the event of default." (Rubin et al., 1999)

> "We told the bankers from the night before to divide themselves into three groups: one to analyze Lehman's toxic assets to help facilitate a potential merger, one to investigate an LTCM-style consortium that could take over the firm and gradually wind down its positions, and one to explore ways to prepare for a bankruptcy and limit the attendant damage." (Geithner, 2015)

Consequently, the rescue was not realized and Lehman Brothers filed for bankruptcy. On the other hand, most of the largest financial institutions at the core of the financial system had been involved in government-assisted or non-assisted rescue mergers[8] to maintain financial stability. As a result, rescue mergers increased the resilience of the financial system against unintended consequences of interconnectedness and counterparty risks[9] in the 2007-2009 period.

In light of the earlier practices, I model bank rescues in a coalition formation framework in the presence of counterparty risks, which is built on the financial contagion model introduced by Elliott, Golub, and Jackson [31]. In a nutshell, the model is as follows. The financial network is defined over exogenously given proprietary assets, external liabilities, and interbank obligations of each bank. A negative shock hits the proprietary asset of a single[10] bank in the network, which represents a case where banks are affected from the financial shock at different degrees and, hence, are able to contribute to each others' rescue. A bankruptcy occurs when a bank is insolvent, i.e., when its total assets are lower than its total liabilities.[11] Bankruptcies, if not prevented, pose additional costs to the system. Rescues occur in the form of consortia of two or more banks (i.e., mergers or capital injections by consortia). Consortia are formed after the shock hits and before the payments are realized so that they can prevent bankruptcies. The coalition formation game is a simultaneous move game and the solution concept is strong Nash equilibrium. Lastly, the government assistance is designed to assist banks' contributions in rescues.

[8]The mega-mergers in that period are as follows (the acquiring institution(s) - the acquired institution (the date)): RBS, Fortis, and Banco Santander - ABN Amro (October 2007); JP Morgan Chase - Bear Stearns (March 2008); Banco Santander - Alliance&Leicester (July 2008); Bank of America - Merrill Lynch (September 2008); Lloyds - HBOS (September 2008 to January 2009), Wells Fargo - Wachovia (October 2008), BNP Paribas - Fortis (May 2009).

[9]Segoviano and Singh [62] quantify counterparty risks in a sample of financial institutions including JP Morgan Chase, Citibank, Bank of America, Goldman Sachs, Merrill Lynch, Lehman, Morgan Stanley, Credit Suisse, Bear Stearns, Wachovia, and Wells Fargo. Counterparty liabilities of broker-dealers in OTC derivatives market as of March 2008 were as follows (in billion dollars): JP Morgan Chase Bank: 68, Citibank: 126, Bank of America: 29, Goldman Sachs: 104, Merill Lynch: 59, Lehman Brothers: 36, Morgan Stanley: 69, Credit Suisse: 70.

See ECB Report [34] for counterparty risks associated with credit default swaps (CDSs).

[10]The single shock environment is enough to reveal the insights into the rescues in the presence of counterparty risks. One can think about significant drops in a given bank's asset returns compared to other banks. On the other hand, multiple shocks environment is closely related to the common asset holdings case where multiple banks' assets deteriorate simultaneously.

[11]Incorporating liquidity driven defaults into the model while considering additional channels such as potential contagious bank runs is a future research direction.

The objective of this paper is twofold. The first objective is to investigate the role of the architecture of the financial network in ultimate failures while taking into account the endogenous rescues. The second objective is to provide interbank exposure structures that minimize government bailouts and, thus, the burden on the taxpayers. I study government assistance in a framework where capital transfers between banks are more efficient than government bailouts. In this framework, networks that minimize the total amount of bailout transfers are welfare-maximizing.

Consequently, there are three sets of results of this study. The main results of the paper show that:

- contagion risk does not necessarily imply financial instability and, surprisingly, leads to greater stability in certain network structures. In such networks, banks take actions against any failure risk and endogenously rescue each other to minimize their losses.

  Accordingly, I characterize networks that minimize bailouts and maximize social welfare and show that:

- welfare-maximizing financial networks are connected through *i)* intermediate levels of interbank liabilities per bank, and *ii)* no clustering of interbank exposures among any subset of banks. In such contagious networks, the losses from any default are (maximally) internalized by banks instead of external creditors (e.g., depositors), which maximize banks contributions in rescues against any distress scenario. As a result, such contagious networks emerge as first-best networks.

More specifically, welfare-maximizing networks are first-best networks that emerge unconditional on the shock level and the location of the initial distress. In such contagious networks, banks more than undo the contagion risk and rescue the initially distressed bank that is hit by the negative financial shock. Contrarily, in financial networks where the counterparty risks are sufficiently low, the contagion risk is eliminated but the failure of a bank hit by a large shock becomes more likely due to the lack of rescue incentives. On the other hand, sufficiently high interbank exposures can cause unavoidable contagion risk even though such high levels of counterparty risks incentivize other banks for rescues. Therefore, intermediate levels of interbank liabilities emerge as one of the main properties of the first-best networks.

In addition, endogenously arising rescues provide novel insights into the *diversification* of interbank exposures, which has been extensively studied in the literature, e.g., Acemoglu, Ozdaglar, and Tahbaz-Salehi [1] and Elliott, Golub, and Jackson [31]. Although the diversification plays a crucial role in *potential* systemic risk and contagiousness, the results show that the ultimate failures do not necessarily depend on the diversification level. Instead, financial stability is determined by two key properties: *i)* the potential bankruptcy losses internalized by banks and *ii)* banks' aggregate capital that determines the loss absorption capacity of the system.

In line with these main results, the second set of results show that inefficiencies occur in certain network structures. Inefficiencies here refer to cases where some failures are not

prevented even though they are preventable via contributions of other banks. There are two main sources of such inefficiencies: the overall structure of the interbank exposures and the level of interbank obligations per bank. One example of inefficiencies is about the concentration of interbank exposures among some groups of banks (i.e., clustering). In such a case, concentration of interbank exposures poses a local contagion risk. Moreover, banks that are less affected from the potential losses have limited rescue incentives. As a result, sufficiently large shocks lead to non-prevented local contagion whenever the banks in the contagious cluster have inadequate capital to rescue each other.

The third set of results show that the government's rescue decisions are selective. For large shocks, the government assists in the rescue of the rest of the system to maintain financial stability instead of assisting in the rescue of the initially distressed bank. In such a case, initial failure occurs and the remaining banks absorb the (relatively low) losses with the government assistance, instead of rescuing the bank hit by a large shock. The results together offer plausible explanations as to why some of the systemically important financial institutions (SIFIs) had not been saved, but many mega-mergers among other SIFIs had been formed with or without government assistance during 2007-2009 period.

Policy implications and historical debate on rescues— Although the global financial system evolves over time, the debate on financial stability and the rescue of distressed banks goes back to Thornton [63] and Bagehot [10].[12]

In his famous book "Lombard Street: A Description of the Money Market", Bagehot explained the importance of timely actions for rescues.[13] The results in this paper provide insights into the historical debate on bank rescues. Timely rescues are important for preventing a potential systemic risk. Furthermore, reserves must be lent to banks "*whenever the security is good*" in Bagehot's words, which corresponds to sufficiently small financial shocks in this framework. Besides, in the presence of counterparty risks, which is one of the major consequences of the interconnectedness of today's complex financial system, the incentives to save distressed banks can be extended to non-crisis circumstances as well.

In addition, the findings help us better understand the implications of current interbank regulations. First, the results suggest that the limitations on common asset exposures are more essential than limitations on interbank obligations if ex-ante moral hazard issues are more prominent than ex-post issues about banks' contributions in rescues. While common-asset exposures limit banks' capabilities to contribute to each others' rescue, counterparty

[12] "...If any one bank fails, a general run on the neighboring ones is apt to take place, which if not checked at the beginning by a pouring into the circulation a large quantity of gold, leads to very extensive mischief." (Thornton, 1802, p. 113)

[13] "...A panic, in a word, is a species of neuralgia, and according to the rules of science you must not starve it. The holders of the cash reserve must be ready not only to keep it for their own liabilities, but to advance it most freely for the liabilities of others. They must lend to merchants, to minor bankers, to "this man and that man", whenever the security is good. In wild periods of alarm, one failure makes many, and the best way to avoid the derivative failures is to arrest the primary failure which caused them." (Bagehot, 1873, pp. 51–2)

risks incentivize banks to rescue each other to avoid excessive losses and orderly defaults. Secondly, the results provide explanations for the potential consequences of clustering of interbank exposures, which is helpful for understanding the implications of the interbank regulations defined in Basel III framework that focus on restricting the level of exposures to a single counterparty or a group of connected counterparties (BIS, [13]). Thirdly, the results support the importance of the regulations on capital requirements, because not only it makes each bank more resilient against shocks but also enables banks to transfer resources to each other, whenever required.

Related literature — Besides the extensive literature on the sources and consequences of systemic risk, models on private and public rescues to avoid bank failures are limited. Common asset holdings and correlated portfolio choices that lead to excessive risk taking and moral hazard issues have attracted considerable attention in the literature. Among others, Acharya and Yorulmazer [3], Gale and Vives [39], Erol [32], and Dávila and Walther [21] focus on such moral hazard issues. On the other hand, the studies on bank rescues in the presence of counterparty risks are limited. Erol [32] builds a model of network formation and studies the moral hazard problem in financial networks. Leitner [54] and Rogers and Veraart [59] provide the initial steps towards understanding the private rescue mechanism. Leitner [54] focuses on the ex-ante optimal size of clusters in a bail-in model. Rogers and Veraart [59] show that the incentives for bank rescues emerge in the presence of bankruptcy costs and provide results for canonical network structures. Proposition 1 and Proposition 2 in this study correspond to the findings of Rogers and Veraart [59] and Leitner [54], respectively. Bernard, Capponi, and Stiglitz [15], in a simultaneous and independent work to mine, cover some of the basic questions here with a different model.

Most importantly, different from studies above, I study bank rescues *i)* under no restrictions on the network structure, *ii)* under no restrictions on coalition formation, *iii)* for heterogenous values of proprietary assets and external liabilities of banks, and *iv)* with/without government assistance. Analyzing rescues in a generalized framework provide novel insights into the problem such as selective rescue decisions, implications of different levels of interbank liabilities (strong vs. weak connectivity), and the importance of the overall network structure (non-clustering). Accordingly, I provide a characterization result for welfare-maximizing networks and show the properties of the first-best networks.

The other specific differences are as follows. The coalition formation nature of the problem alter the earlier results in Leitner [54]. Leitner [54] shows that an optimal network for given asset realizations might be non-optimal for some other realization of asset returns. Different from Leitner [54], I show that welfare-maximizing networks emerge unconditional on the source and the magnitude of the shock. Bernard, Capponi, and Stiglitz [15] compare banks' contributions in ring and complete networks while considering whether a government's threat not to bailout is credible or not. As a result, their approach focuses on incentive problems under a different government intervention method, which is another way of modeling government intervention under perfect information. Similar to Leitner [54], Bernard et al.

[15] also focuses on the rescue of the shocked bank only, whereas the coalition formation framework in this paper provides a systematic approach to rescue formation.

The remainder of the paper is organized as follows. In Section 2 and Section 3, I introduce the network model and rescue formation, respectively. Section 4 includes the analyses on sources of inefficiencies in rescues. In Section 5, I study the welfare-maximizing networks and government assistance. Section 6 concludes.

2 A Model of Interconnectivity, Counterparty Risks, and Contagion

2.1 Assets, Liabilities, and the Interbank Network

In this part, I introduce the interbank network model that I use throughout the paper, which is a version of the financial network model introduced by Elliot, Golub, and Jackson [31]. There is a set $N = \{1, ..., n\}$ of banks. Each bank i is endowed with an exogenously given proprietary asset p_i that generates cash flow (e.g., interest-earning loans). In addition, each bank is endowed with exogenously given external and interbank liabilities. The external liabilities of each bank i is denoted by l_i and might be thought of as its obligations to external creditors (e.g., depositors) or other obligations such as operational expenses (e.g., wages or tax). The interbank obligations among banks are represented as claims that banks hold in each other. For exogenously given $0 < C_{ij} < 1$, bank i is a creditor of bank j, and bank i claims C_{ij} portion of the total assets of bank j when payments are realized.[14] The claims are such that $C_{ii} = 0$ for all $i \in N$ (a bank holds no interbank claims in itself), $C_{ji} \geq 0$ for all $i \neq j$ and $\sum_{j \in N} C_{ji} < 1$ for all $i \in N$. The interdependencies among banks through interbank obligations can be represented as a weighted directed graph, where the C matrix is an $n \times n$ matrix called the *claims matrix*. Finally, each bank i is owned by a single distinct shareholder such that each shareholder only holds the shares of a single bank.[15]

Given these specifications of the model, a financial network is represented by (C, F) where C is the claims matrix representing the network characteristics and F is the bank characteristics including the information on the proprietary assets, the external liabilities, and lastly the bankruptcy costs that I discuss next.

[14] The way of modeling the interbank contracts here is different than the standard way of modeling the debt contracts among banks. The claims represented in ratios might be thought of as a mapping from "face values of debt contracts" to "ratios of the face values of debt contracts to total assets". Then, the only difference from the standard way of modeling is that the claims create linearities in interbank contracts, which provide a tractable model of contagion and have no further implications on results except level effects. Figure 13 in the Appendix depicts the interdependencies in balance sheets.

[15] This implies that there exist no cross-equity holdings and, hence, no conflict of interest in rescue decisions among the shareholders of banks.

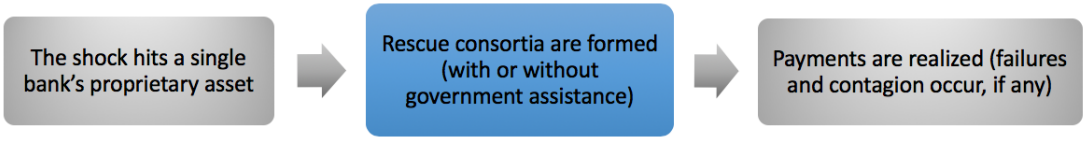

Figure 1: Timing of events

2.2 Financial Shock, Payment Equilibrium, and Potential Financial Contagion

As illustrated in Figure 1, the timing of events[16] is as follows.

An exogenously given negative shock $s \in [0, p_r]$ hits the proprietary asset of a single bank, denoted by bank r. Banks form rescue consortia to prevent potential failure(s), which occurs before the payments are realized. After the rescue formation, payments to both external and internal creditors are realized simultaneously.

In the rest of Section 2, I introduce the financial contagion framework that is built on Elliot, Golub, and Jackson [31] and explain how financial contagion would occur if there were no rescue attempts, before introducing the rescue formation in Section 3.

Payment equilibrium captures the bankruptcy situation as follows. When the payments are realized, any bank that has total assets less than total liabilities becomes insolvent and bankruptcy (or default) occurs.[17] Bankruptcy is costly. The proprietary asset of a bank drops at some rate in case of bankruptcy. Exogenously given β_i captures the bankruptcy cost of bank i in nominal terms and ζ_i captures the ratio of bankruptcy cost of bank i to its proprietary assets. In other words, $1 - \zeta_i$ is the recovery rate of the asset's full value in a bankruptcy situation. Correspondingly, the bank hit by the negative shock has a bankruptcy cost equal to $\beta_r = (p_r - s)\zeta_r$. Any other bank $i \neq r$ has a bankruptcy cost equal to $\beta_i = p_i\zeta_i$.

If there is any such bank i that defaults, then the shareholder of bank i is wiped out and receives nothing, whereas both external and internal creditors of such bank i are rationed in proportion to total assets of bank i with equal seniority.

By incorporating the bankruptcy condition (with no rescue attempts), the total assets of bank $i \neq r$ when the payments are realized is given by:

$$V_i = (\sum_{j \in N} C_{ij} V_j) + (p_i - b_i) \tag{1}$$

which is equal to the sum of its interbank assets and the recovery value of its proprietary asset, where $b_i = \beta_i$ if bank i defaults and 0 otherwise. Similarly, the total assets of the bank hit by the shock is given by:

[16] In the model, there is no specific notation used for the timing of events. Figure 1 provides a useful picture of the ordering of the events, which is as implied by the model.

[17] A bankruptcy in the model refers to an insolvency case.

$$V_r = (\sum_{j \in N} C_{rj} V_j) + (p_r - s - b_r) \tag{2}$$

In matrix notation:

$$\mathbf{V} = (I - C)^{-1}(\mathbf{p} - \mathbf{b}) \tag{3}$$

where $\mathbf{p} = [p_1, ..., p_r - s, ..., p_n]'$ is the vector of proprietary assets after the shock, and $\mathbf{b}$ is the vector of realized bankruptcy costs such that $b_i = \beta_i$ if bank i defaults and 0 otherwise. The total liabilities of bank i is equal to the sum of its interbank and external liabilities, which is given by:

$$L_i = (\sum_{j \in N} C_{ji} V_i) + l_i \tag{4}$$

Lastly, by incorporating the bankruptcy condition, shareholders' equity (or net worth) of bank i is equal to:

$$e_i = max\{0, V_i - L_i\} \tag{5}$$

Following equations (4) and (5), shareholders' equity can be rewritten as:

$$e_i = max\{0, v_i - l_i\} \tag{6}$$

where $v_i = \hat{C}_{ii} V_i$ and $\hat{C}_{ii} := 1 - \sum_{j \in N} C_{ji} > 0$ is the portion of the total assets of bank i which is not claimed by other banks in the network.[18] The vector $\mathbf{v}$ can be rewritten in matrix notation as follows:

$$\mathbf{v} = \hat{C}(I - C)^{-1}(\mathbf{p} - \mathbf{b})$$

$$\mathbf{v} = A(\mathbf{p} - \mathbf{b}) \tag{7}$$

$A = \hat{C}(I - C)^{-1}$ matrix is a column stochastic matrix, called the dependency matrix. As a result, the shareholders' equity and the payments to internal and external creditors of each bank are determined simultaneously via the payment solution satisfying Equations (3) to (7). There always exists a payment solution and there can be multiple solutions.[19] The contagion algorithm below gives the ultimate failures when the payments are realized. As in Elliott et al. [31], the contagion algorithm is based on the best-case solution in which as few banks as possible fail. The algorithm works as follows:

[18] $\hat{C}_{ii}$ is assumed to be strictly positive. In matrix notation, $\hat{C}$ is an $n \times n$ diagonal matrix such that $\hat{C}_{ii} > 0 \ \forall i$ and $\hat{C}_{ij} = 0 \ \forall i \neq j$. By this assumption, the inverse $(I - C)^{-1}$ is well defined and non-negative.

[19] See Elliott et al. [31] for a detailed discussion on the existence and the multiplicity of payment solution. There exists two main sources of the multiplicity of the payment solution. One source is based on the story of self-fulfilling failures as in the Diamond and Dybvig [24] model, and the other source of the multiple solution is based on the interdependencies in the financial network.

At step t of the algorithm, let $\aleph_t$ be the set of failed banks. Initialize $\aleph_0 = \emptyset$.[20] At step $t \geq 1$:

(i) Let b_{t-1} be a vector with element $b_i = \beta_i$ if $i \in \aleph_{t-1}$ and $b_i = 0$ otherwise.

(ii) Let $\aleph_t$ be the set of all j such that entry j of the following vector is negative:

$$A[\mathbf{p} - \mathbf{b}_{t-1}] - l$$

(iii) Terminate if $\aleph_t = \aleph_{t-1}$. Otherwise return to step 1.

When the algorithm terminates at step T, the set $\aleph_T$ corresponds to the set of banks that fail in the best-case solution. The algorithm provides us the domino failures in the network, where various banks default at each step, which are triggered by the defaults in the previous steps. By using this algorithm one can find the set of failures for a given financial network (C, F). This hierarchical default structure works for any given financial network.

Next, I provide definitions on banks and networks and show simplified illustrations for financial contagion.

DEFINITION **1** *Bank i is a distressed bank if $i \in \aleph_T$, and it is a healthy bank if $i \in N \setminus \aleph_T$.*

Following Definition 1, any bank that would default if there were no rescue attempts in a given financial network is called a *distressed bank*, and any bank that would not default if there were no rescue attempts is called a *healthy bank*.

Definition 2 is about the contagiousness of a financial network (C, F). A financial network (C, F) is *potentially contagious* if both the shocked bank (bank r) and its creditors (d_r^{out}) are distressed banks. Formally,

DEFINITION **2** *Consider a given financial network (C, F) and shock s hitting bank r in (C, F). Then,*

i) (C, F) is potentially contagious if $\{r \cup d_r^{out}\} \subseteq \aleph_T$, where d_r^{out} is the set of creditors of bank r and $\aleph_T$ is the set of distressed banks in (C, F). Any potential failure $i \in \{\aleph_T \setminus r\}$ is called a potential contagious failure.

ii) (C, F) is non-potentially contagious if $\aleph_T \subseteq \{r\}$ (either there exists no distressed bank or only bank r is a distressed bank).

In a potentially contagious network, there might be additional potential failures than the shocked bank and its creditors, due to the domino effects of financial contagion. Potential contagion scenarios are illustrated in Figure 2.[21] A link emanating from bank i directed to bank j represents the case that bank j is the creditor of bank i, and there is a flow from bank i to bank j when the payments are realized. One can imagine that there exists no

[20] For $s = 0$, it is considered that a given financial network (C, F) is such that all banks are always able to pay back their external and interbank liabilities in full and the net worth of each bank is non-negative.

[21] The bank and the network characteristics are not specified in the illustration in Figure 2. A specific numerical example can always be written for given potential contagion cases.

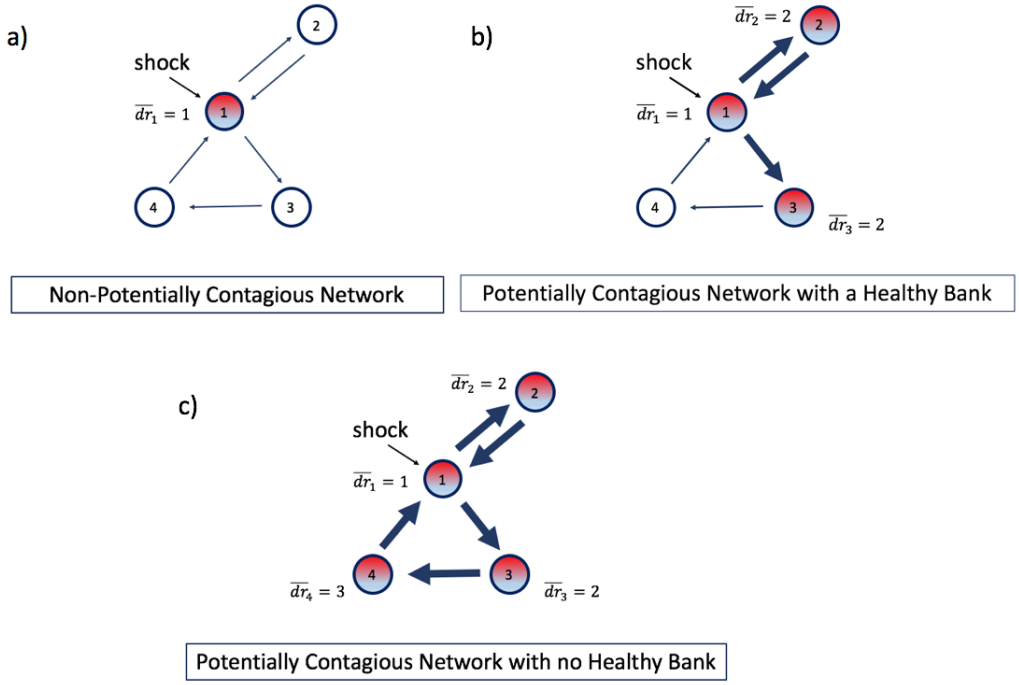

Figure 2: Illustrations of potential contagion scenarios

potential contagious failure if the interbank liabilities among banks are sufficiently small, which is represented in Figure 2.a. Levels of interbank liabilities, level of the shock, network architecture and bank characteristics all together determine the extent of potential contagion as illustrated in Figure 2.b and 2.c.

3 Rescue Formation

Coalitions among banks are formed after the shock hits but before the payments are realized. Therefore, coalitions that are formed before the shock propagates into the system can avoid costly bankruptcies. A rescue consortium might include two banks, referring to a rescue merger, or more than two banks, referring to a multi-bank rescue consortium. In the rest of the paper, I use the terms consortium and merger interchangeably. The coalition formation framework allows for formation of multiple rescue mergers, where each bank is involved in at most one merger.

A merger is defined in a standard way. The total assets and total liabilities of the members of a given merger are summed up, and unchanged for the non-merged banks.[22] A consortium might be thought of as pooling the available capital for rescues. For instance,

[22]Formation of mergers requires restructuring of the *claims matrix*. Lemma 2 in the Appendix shows that there exists a unique way of restructuring the claims which satisfies the properties in Definition 3 for every given proprietary asset return vector $\mathbf{p}$. The restructured claims are set according to the result in Lemma 2.

the formation of the grand coalition does not represent a case where all banks become a single bank; rather, it represents the case in which all banks contribute to the rescue of the failure of the shocked bank at some degrees. This is equivalent to capital injections by other banks to the bank hit by a financial shock. Formally, $M = \{m_1, ..., m_n\}$ is the set of rescue consortia formed and $\phi \subseteq N$ is the set of banks that are involved in any rescue consortium. The financial network after the mergers[23] are formed is defined as follows:

DEFINITION **3** *$(C, F)^M$ is the financial network after a set of banks $\phi \subseteq N$ form the set of mergers $M = \{m_1, ..., m_n\}$ in a given financial network (C, F). $(C, F)^M$ has the following properties:*

i) $N^M = (N \setminus \phi) \cup M$,

ii) $p_j^M = p_j$, $l_j^M = l_j$, $\beta_j^M = \beta_j$ $\forall j \in N \setminus \phi$,

iii) $V_j^M = V_j$, $L_j^M = L_j$ $\forall j \in N \setminus \phi$,

iv) $p_{m_k}^M = \sum_{k \in m_k} p_k$, $l_{m_k}^M = \sum_{k \in m_k} l_k$, $\beta_{m_k}^M = \sum_{k \in m_k} \beta_k$ $\forall m_k \in M$,

v) $V_{m_k}^M = \sum_{k \in m_k} V_k$, $L_{m_k}^M = \sum_{k \in m_k} L_k$ $\forall m_k \in M$.

Next, I define *rescue merger*. A merger $m_k \in M$ is a *rescue merger* if it prevents at least one additional failure compared to the case that m_k has not been formed, all else constant. Formally,

DEFINITION **4** *A merger $m_k \in M$ is a rescue merger if $\aleph_T^M \subsetneq \aleph_T^{M \setminus m_k}$, where $\aleph_T^M$ and $\aleph_T^{M \setminus m_k}$ are the sets of distressed banks in $(C, F)^M$ and $(C, F)^{M \setminus m_k}$, respectively.*

Next, I define the social welfare and socially efficient set of mergers.

DEFINITION **5** *The social welfare in $(C, F)^M$ is equal to $W(C, F)^M = \sum_{i \in N^M} (p_i - b_i) - s$ where $b_i = \beta_i$ if bank $i \in N^M$ defaults, and 0 otherwise.*

In this framework, maximizing the social welfare is equivalent to minimizing the sum of the realized bankruptcy costs since the losses are the shock that is irreversible and the sum of

Besides its uniqueness property, the restructuring rule given in Lemma 2 is a natural way of restructuring the interbank obligations.

[23] Mergers are formed after the shock hits before the payments are realized. In this framework, it means that mergers are formed after a bank that is known by every other bank is hit by the shock but before any payments are realized and, so, before the shock propagates into the system. Therefore, the given definition of mergers that does not take the shock into account is the appropriate way of defining the pooling of available capital for rescues. As a result, $\mathbf{p^M}$ in Definition 3 does not capture the shock. However, in order to avoid heavy notation, I used the same superscript as in Definition 3 for proprietary assets, total assets and total liabilities (p^M, V^M, L^M) for payment realization after the mergers are formed, which captures the shock and the realized bankruptcy costs as well.

the realized bankruptcy costs that might be reduced or totally eliminated via rescues. For any given financial network (C, F), a set of mergers M is socially efficient if it maximizes the social welfare. Formally,

DEFINITION **6** *A set of mergers M is socially efficient if $\nexists\, M'$ s.t. $W(C,F)^{M'} > W(C,F)^{M}$.*

3.1 Rescue Formation Game

Informally, the game can be explained as follows. Mergers are formed under perfect information. Before the shock and payments are realized, each bank announces the mergers in which it accepts to be involved. Given all strategies, mergers are formed simultaneously where each bank is involved in at most one merger. Given the sharing rule, banks share the payoff of the coalition. Sharing the payoff of a coalition can be thought of as sharing the contributions in rescues. Then, the economy following the formation of mergers is a partition of the initial economy, now consisting of rescue consortia and the remaining banks. A partition is an equilibrium outcome if there exists no coalition that deviates where each bank in the deviating coalition receives weakly higher payoff after deviation.

Formally, the rescue formation game is a simultaneous move game $\Gamma = ((S_i)_{i\in N}; (f_i)_{i\in N})$ consisting of set of banks $N = \{1, ..., n\}$, a strategy set S_i for each bank $i \in N$, and a payoff function $f_i : \prod_{i\in N} S_i \to \mathbb{R}$ for each bank $i \in N$. A particular strategy $s_i \in S_i$ represents the set of mergers that bank i has willingness to be involved. The strategy set of bank i is $S_i = P(\{T \cup i \mid T \subseteq P(N \setminus i)\})$ where $P(N \setminus i)$ is the set of subsets of $N \setminus i$ and $P(\{T \cup i \mid T \subseteq P(N \setminus i)\})$ is the set of subsets of $\{T \cup i \mid T \subseteq P(N \setminus i)\}$. Given the strategy profiles, a bank is involved in at most one merger. A merger m_k is formed if $\{m_k\} \subseteq s_j$ for all $j \in m_k$ and there exists no other m_l and $j \in m_k, m_l$ s.t. $\{m_l\} \subseteq s_i$ for all $i \in m_l$. If there exist such ties, then the ties are broken in a way that the social welfare is maximized.

The solution concept is the strong Nash equilibrium (SNE)[24], which requires stability against deviations by every conceivable coalition. Thus, an equilibrium is strong if there exists no coalition, taking the actions of its complement as given, that can deviate in a way that benefits all the members of the coalition. Formally, $s^* \in \prod_{i\in N} S_i$ is a SNE if and only if $\forall G \subseteq N$ and $\forall s_G \in \prod_{i\in G} S_i$, there exists an agent $i \in G$ such that $f_i(s^*) \geq f_i(s_G, s^*_{N\setminus G})$.

Next, I define payoffs. For given $(C, F)^M$ and the sharing rule, each bank $i \in N$ receives a payoff denoted by e_i^M.

First, I define a class of sharing rules Δ. In summary, the class of sharing rules below guarantees that there exists no unintuitive sharing rule that would cause technical issues in the analysis. More specifically, the property below ensures that if a collective action taken by a set of agents increase the total payoffs of these agents, then all agents in this group should weekly benefit from such an action. It is similar to the monotonicity property in

[24]See Section 7.1 in the Appendix for the discussion on coalition formation in a cooperative game setting

Kalai-Smorodinsky bargaining solution, but since there are externalities and the payoffs are based on the partition of the economy, I provide the Definition 7.

DEFINITION **7** *A class of sharing rules Δ is such that any sharing rule $\delta \in \Delta$ satisfies the following properties:*

i) $f_i(s_i, s_{-i}) = \alpha_i^{m_k} e_{m_k}^M$ where $\sum_{i \in m_k} \alpha_i^{m_k} = 1$ for all $m_k \in M$ (equivalently $\sum_{i \in m_k} f_i(s_i, s_{-i}) = e_{m_k}^M$ for all $m_k \in M$),

ii) Consider any given three partitions M,M' and M'' of a given economy and consider mergers $m_j \in M''$, $m_k \in M'$, $m_l \in M$, where $m_k \cup m_l = m_j$. If $e_{m_j}^{M''} \geq e_{m_k}^{M'} + e_{m_l}^{M}$, then $f_i^{m_j}(M'') \geq f_i^{m_k}(M')$ for all $i \in m_j \cap m_k$ and $f_i^{m_j}(M'') \geq f_i^{m_j}(M)$ for all $i \in m_j \cap m_l$. This holds for any such triple $((m_l, M), (m_k, M'), (m_j, M''))$ for any given economy.

Any sharing rule having properties described above is such that i) the total payoff of a coalition is shared among its members, and ii) for everything else constant, if the superset of some coalitions results in a weakly higher payoff than their separate sum, then any member involved in one of these coalitions receives weakly higher individual payoff if the superset is formed.

I choose any $\delta \in \Delta$ and fix it as the sharing rule. For given set of mergers, the contagion algorithm introduced in 2.2 applies for finding the set of failures in the network $(C, F)^M$ after the mergers are formed.

3.2 Bankruptcy Costs

Before moving to network structure related inefficiencies, first, I discuss the role of bankruptcy costs in rescue decisions. Rogers and Veraart [59] show that the existence of bankruptcy costs is crucial for rescues. The same result applies here as well, which is given in Proposition 1 below.

PROPOSITION **1** *Consider a financial network (C, F) such that the bankruptcy costs are equal to zero ($\zeta_i = 0$ for all $i \in N$), the shocked bank (bank r) has high external liabilities (there exists an l_r^* such that l_r is greater than the threshold level l_r^*), and there exists at least one distressed bank in (C, F). Then, there exists no strong Nash equilibrium in which any bank is involved in a rescue merger, and there exists a strong Nash equilibrium in which no merger has been formed.*

Proposition 1[25] shows that whenever bankruptcy costs are zero, potential rescuer banks have no incentives to form rescue mergers. The reason is that any rescue that does not prevent emergence of bankruptcy costs is unprofitable for rescuers because rescuing a distressed

[25] $l_r > l_r^*$ is a condition that rules out formation of some unintuitive mergers, where a set of banks affect the propagation of the initial shock into the system via formation of a merger that includes bank r.

bank is a costly action since total assets of a distressed bank is lower than its total liabilities. Following this initial result, I provide a simplifying assumption.

Assumption 1. $\zeta_i = 1$ (bankruptcy costs are high), $p_i = 1$ and $l_i = l$ for all $i \in N$. (homogeneity assumption, which is relaxed in Section 5)

Assumption 1 is a simplifying assumption implying that the proprietary asset of any single bank or coalition drops to zero if it defaults.[26] Besides, $p_i = 1$ and $l_i = l$ are the other simplifying assumptions for the rest of the analysis in Section 4 and first part of Section 5. I relax this assumption and study heterogeneity in bank sizes and connectivity in the second part of Section 5. Focusing on networks with homogeneity in bank characteristics ensures that any inefficiency is due to the network characteristics rather than the heterogeneity in sizes or leverage ratios of the banks.

Assumption 2. $0 \leq \sum_{j \in N} C_{ji} \leq \frac{1}{2}$ holds for all $i \in N$ for any financial network (C, F).

For the rest of the analysis, I consider that the interbank liabilities of a bank cannot exceed half of its total assets in any given financial network.[27]

4 Network Characteristics and Inefficiencies in Rescues

First, I provide Definition 8 on network characteristics. The diversification and the integration are the two measures of network characteristics.

DEFINITION 8 *i) The financial network (C', F) is more diversified than the financial network (C, F) if and only if*

$C'_{ji} \leq C_{ji}$ for all (i, j) such that $C_{ji} > 0$, and $C'_{ji} < C_{ji}$ for some (i, j),

$C'_{ji} > C_{ji} = 0$ for some (i, j).

ii) The financial network (C', F) is more integrated than the financial network (C, F) if and only if $\sum_{j:j \neq i} C'_{ji} \geq \sum_{j:j \neq i} C_{ji}$ for all i, with strict inequality for some i.

A financial network becomes more diversified when the number of creditors of each bank i weakly increases, where the interbank liabilities of bank i to its each original creditor weakly decrease. Thus, the diversification captures the spread of interbank contracts in a financial

[26] Acemoglu et al. [1] explain this situation as follows: "...Furthermore, during bankruptcy, the liabilities of the institution may be frozen and its creditors may not immediately receive payment, leading to effectively small recovery rates." (Acemoglu et al. [1]).

[27] Assumption 2 is a technical assumption, which guarantees that the shock itself mostly affect the bank hit by the shock and its creditors.

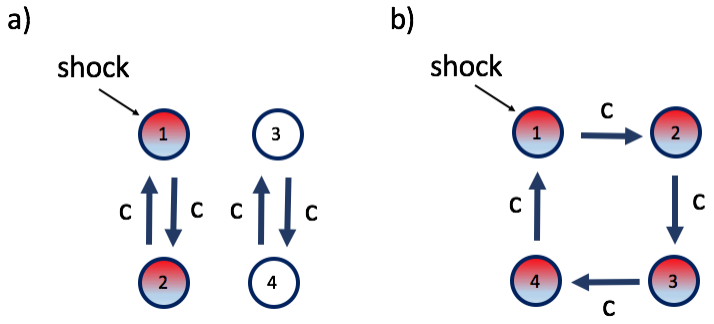

Figure 3: Potential failures in disconnected vs connected networks

network. On the other hand, a financial network becomes more integrated if the ratio of the interbank liabilities to the total assets becomes higher for each bank. This refers to stronger interbank dependencies (or stronger ties) for given set of counterparties for each bank.

4.1 Beyond Bank Level Network Characteristics: The Network Topology

In addition to the bank level network characteristics, the network topology is the other main characteristics of a given network. Figure 3 shows how networks that have different topologies can have different potential contagion risks even though the bank level network characteristics (diversification and integration) are identical in the two networks. In Figure 3, each bank has a single counterparty and the integration level is equal to $\sum_j C_{ji} = c$ for each bank i in both networks. In Figure 3.a, potential contagion is local, whereas in Figure 3.b connectedness creates a system-wide contagion risk.[28]

One source of inefficiencies that arise due to network characteristics is the disconnectedness. For this result, I consider a network consisting of isolated islands where banks in each island form complete connections. Formally,

DEFINITION **9** *A financial network (C, F) is an islands-connected financial network if the network consists of set of $J = \{J_1, ..., J_n\}$ isolated islands where banks in each island form complete connections and banks in different islands have no connections. An islands-connected financial network (C, F) satisfies the following properties:*

[28]The potential contagion in financial networks in Examples 1 and 3 are illustrated in Figure 2.a and 2.b, respectively.

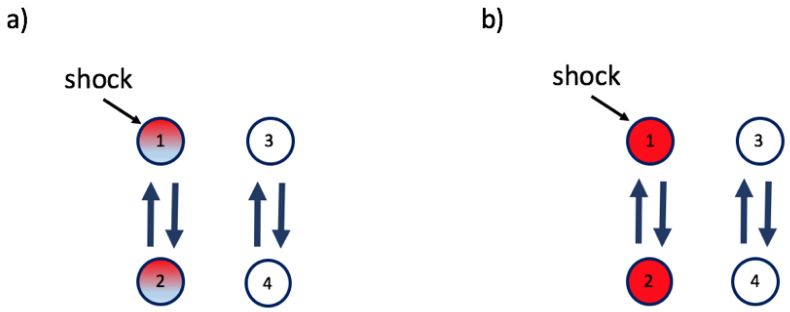

Figure 4: Non-prevented local contagion (Potential failures (panel a) and equilibrium (panel b) in Example 1)

$$(C,F) := \begin{cases} \sum_{k\in N} C_{kl} = \overline{c}_{J_k} & \forall l \in J_k \\ C_{kk} = 0 & \forall k \in N \\ C_{kl} = \frac{\overline{c}_{J_k}}{|J_k|-1} & \forall k, l \neq k \in J_k, \forall J_k \in J \\ C_{kl} = 0 & \forall k \in J_k, l \notin J_k, \forall J_k \in J \end{cases}$$

where $\overline{c}_{J_k}$ *is the integration level of each bank in an isolated island* $J_k \in J$ *, and* $\mid J_k \mid$ *is the number of banks in an isolated island* $J_k \in J$*.*

Example 1 below shows how inefficiencies arise in disconnected networks. In the rest of the paper, I provide five more examples that each of them are a modified version of Example 1. Each example illustrates the findings in the related part and is useful for the comparison of findings in that part with the findings in the previous parts.

EXAMPLE **1** *Consider a financial network* (C,F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$l_i = 0.93 \text{ for all } i \in N$$
$$p_i = 1 \text{ for all } i \in N$$
$$\sum_j C_{ji} = 0.1 \text{ for all } i \in N$$
$$s = 0.25 \text{ hits bank 1}$$

Figure 4 depicts the equilibrium outcome in Example 1, where banks 3 and 4 have no rescue incentives and banks 1 and 2 are not adequately capitalized for rescues. As a result, as represented in Figure 4, a non-prevented local contagion occurs even though there is enough capital in the system to prevent all failures. On the other hand, connectedness might bring system-wide contagion risk as illustrated in Figure 3. The ways to avoid such risks in connected networks are discussed in Section 4.2 and 5.

Next, I provide a result on the *disconnectedness*. Definition 10 is the formal statement of the rescue capabilities in a financial network.

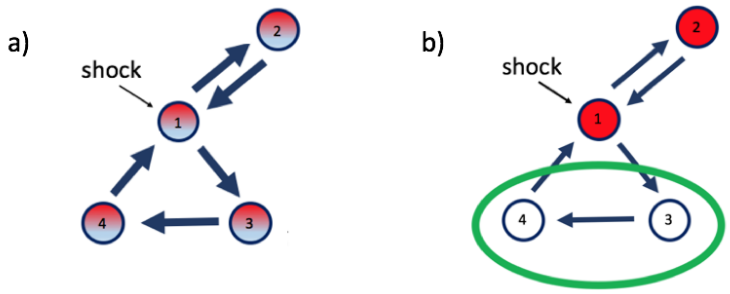

Figure 5: Partially prevented contagion (Potential failures (panel a) and equilibrium (panel b) in Example 2)

DEFINITION **10** *In a given financial network* (C, F)*, banks have the capability to avoid the bankruptcy of a set of distressed banks* $D \subseteq \aleph_T$ *if there exists a set of mergers* M *such that*

i) $i \notin \aleph_T^M$ *for all* $i \in D$*,* $i \notin \phi$ *where* ϕ *is the set of banks that are involved in a merger, and*

ii) $m_k \notin \aleph_T^M$ *for all* $i \in D$*,* $i \in m_k$*,* $m_k \in N^M$*.*

PROPOSITION **2** *Consider an islands-connected financial network* (C, F) *where the bank hit by the shock is located in the isolated island* J_r*. Consider also that banks in* (C, F) *have the capability to prevent the bankruptcy of any subset of the set of distressed banks. Then, for* $s > \mid J_r \mid (1-l)$ *and* $\overline{c}_{J_r} > c^*(s, l, \mid J_r \mid)$*, there exists no strong Nash equilibrium in which any distressed bank is involved in a rescue merger, and there exists a strong Nash equilibrium in which no merger has been formed. Consequently, a non-prevented local contagion occurs, where all banks in the contagious island default.*

As a result, capacity constraints in the contagious region and lack of rescue incentives of the unaffected banks lead to local unavoided contagion in disconnected networks (or islands-connected networks).

Similar to the disconnected case, high concentration of interbank exposures among some group of banks is another source of inefficiencies. Example 2 below shows how concentration of exposures of banks 1 and 2 results in *partially prevented contagion*. Bank characteristics, integration levels and the shock level in Example 2 are the same as in Example 1.

EXAMPLE **2** *Consider a financial network* (C, F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0.1 & 0 & 0.1 \\ 0.05 & 0 & 0 & 0 \\ 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$l_i = 0.93 \text{ for all } i \in N$$
$$p_i = 1 \text{ for all } i \in N$$
$$\sum_j C_{ji} = 0.1 \text{ for all } i \in N$$
$$s = 0.25 \text{ hits bank 1}$$

In Example 2, there is a system-wide contagion risk as shown in Figure 5.a. The payoffs for different merger configurations are given in Table 1 in the Appendix. Different from the

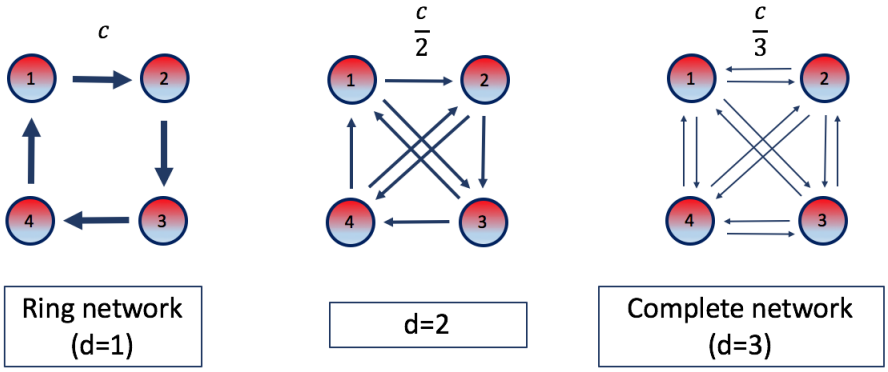

Figure 6: A sequence of $d-$ring lattices with four banks

case in Example 1, here banks form a rescue merger, yet the merger prevents the potential failures partially. The main insight from Example 2 is that the losses due to failure of banks 1 and 2 are mostly borne by external creditors of these two banks and are not internalized by banks 3 and 4. There exists no strong Nash equilibrium in which bank 1 or bank 2 is rescued. The only merger that can rescue bank 1 is the grand coalition. However, it cannot be an equilibrium outcome since banks 3 and 4 deviate and receive a larger payoff. Moreover, the two mergers that can prevent the failure of bank 2 are the grand coalition and the merger of banks 2, 3, and 4. In both cases, banks 3 and 4 deviate and receive a larger payoff. On the other hand, in any equilibrium, banks 3 and 4 form a coalition and absorb the losses from the failures of banks 1 and 2. This example shows that banks can prefer preventing failures partially, which represents another source of inefficiencies.

Next, I define a network structure in which banks form non-clustered connections as opposed to the disconnected/clustered networks cases.

DEFINITION **11** *i) A d-ring lattice* $(\overline{c}, d, \overline{F})$ *is an evenly-connected network with properties below, where* $i - \{k\}_{k \leq N-1}$ *refers to the* k^{th} *node preceding the node* i *when the nodes are ordered on a circle.*

$$(c, d, F) := \begin{cases} C_{ii} = 0 & \forall i \in N \\ C_{ij} = \frac{c}{d} & \forall \{i, j : i - d \leq j \leq i - 1\} \\ C_{ij} = 0 & otherwise \end{cases}$$

ii) $\psi^R(\overline{c}, \overline{F})$ *is a class of d-ring lattices where a given network* $(\overline{c}, d, \overline{F}) \in \psi^R(\overline{c}, \overline{F})$ *has integration level* $\overline{c}$*, diversification level* d*, and bank characteristics* $\overline{F}$*. The networks in* $\psi^R(\overline{c}, \overline{F})$ *only differ in diversification level, all else equal.*

Figure 6 is an illustration of a sequence of d-ring lattices.

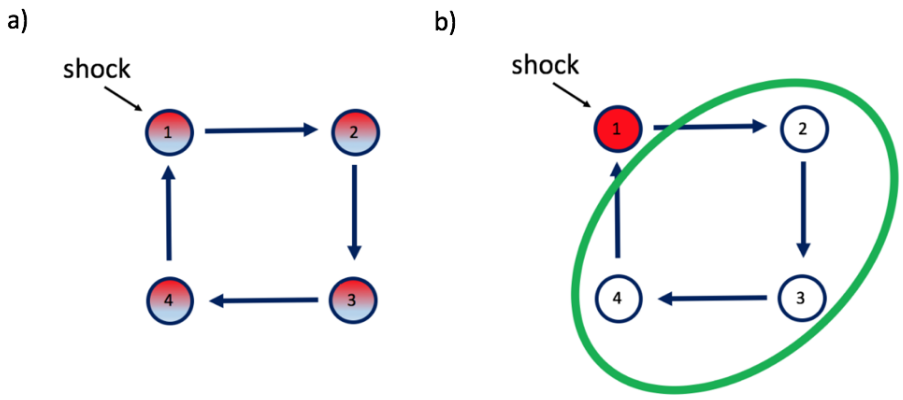

Figure 7: Non-concentrated interbank exposures (no clustering) solve partially avoided contagion issue (Potential failures (panel a) and equilibrium (panel b) in Example 3)

In a given d-ring lattice $(\overline{c}, d, \overline{F})$, the ratio of the interbank liabilities to the total assets is equal to $\overline{c} = \sum_{j \in N} C_{ji}$ for each bank $i \in N$. Moreover, each bank is the creditor of the closest d number of preceding banks and the borrower of the closest d number of following banks in any given $(\overline{c}, d, \overline{F})$. In addition, $C_{ij} = \frac{\overline{c}}{d}$ for all (i, j) such that $C_{ij} > 0$, which capture the homogeneity in the weights of the links. For a given d-ring lattice $(\overline{c}, d, \overline{F})$, the diversification is increasing in d for given integration $\overline{c}$. The diversification level of $d = 1$ refers to the ring network in which each bank i is the single borrower of the bank following bank i, and $d = N - 1$ refers to the complete network in which every pair of distinct nodes is connected by a pair of links.

Example 3 illustrates an evenly-connected network. Bank characteristics, integration level and shock level in Example 3 are same as in previous examples. The only difference here is the formation of linkages.

EXAMPLE **3** *Consider a financial network* (C, F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0.1 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$l_i = 0.93 \text{ for all } i \in N$$
$$p_i = 1 \text{ for all } i \in N$$
$$\sum_j C_{ji} = 0.1 \text{ for all } i \in N$$
$$s = 0.25 \text{ hits bank 1}$$

As shown in Figure 7, evenly-connectedness solve *partially prevented contagion* issue. The main insight here is that banks internalize sufficiently high amounts of potential bankruptcy losses from each potential contagious failure. In that case, banks prefer to stop all *potential contagious failures* as long as it is feasible.

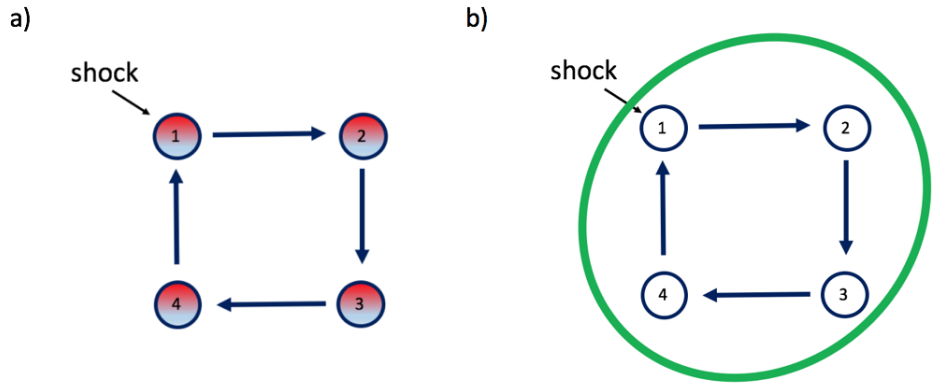

Figure 8: Sufficiently high integration and evenly-connectedness maximize rescue incentives (Potential failures (panel a) and equilibrium (panel b) in Example 4)

4.2 The Levels of Interbank Exposures

4.2.1 Low Levels of Interbank Liabilities

In addition to the disconnectedness and non-evenly-connectedness, another source of inefficiencies is weak interbank ties (low integration). As shown in Figure 7, although non-concentrated exposures solve the partially avoided contagion issue, the shocked bank is still non-rescued in this example. In this part, I explain the ways to avoid such inefficiencies.

The insight here is that for sufficiently high shocks, preventing the failure of the shocked bank might be the second-best solution for the remaining banks. This might still be the case even if there exists a potential cascade of failures that threatens all banks in the network. In that case, remaining banks use the alternative rescue method: rescuing the rest of the system instead of rescuing the shocked bank. This alternative rescue method is illustrated in the equilibrium of Example 3 in Figure 7.b. Next, I provide Example 4 below, in which banks have a higher level of integration with all else the same as in previous examples.

EXAMPLE **4** *Consider a financial network* (C, F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$$

$$l_i = 0.93 \text{ for all } i \in N$$
$$p_i = 1 \text{ for all } i \in N$$
$$\sum_j C_{ji} = 0.2 \text{ for all } i \in N$$
$$s = 0.25 \text{ hits bank 1}$$

Figure 8 depicts how sufficiently high integration solves the issue of non-rescue of the shocked bank. Accordingly, Theorem 1 is the first main result.

THEOREM **1** *Consider a d-ring lattice* $(\overline{c}, d, \overline{F})$ *such that* $1 - l < \frac{\overline{c}(1-\overline{c})}{d}$ *(which guarantees that* $(\overline{c}, d, \overline{F})$ *is potentially contagious) where the banks in* $(\overline{c}, d, \overline{F})$ *have the capabilities to*

prevent all potential failures and have the capabilities to prevent "the potential contagious failures in full" even if the shocked bank defaults. Then, all potential contagious failures are prevented in any strong Nash equilibrium, and there exists a threshold level of interbank claims $c^*(s)$ such that

i) for $\overline{c} > c^*(s)$, the unique strong Nash equilibrium outcome is such that all potential failures, including the failure of the shocked bank, are prevented, and

ii) for $\overline{c} < c^*(s)$, the unique strong Nash equilibrium outcome is such that the shocked bank is not rescued but "potential contagious failures" are prevented in full,

where the minimum level of interbank liabilities that is required to incentivize other banks to rescue the shocked bank is increasing in the shock s.

Theorem 1 has two interpretations.

First, Theorem 1 shows how non-concentrated (non-clustered) interbank exposures eliminate the inefficiencies that would arise in disconnected or clustered networks, as discussed earlier. In such evenly connected networks, banks maximally internalize the potential losses that would arise from any failure.

Secondly, it explains the inefficiencies that arise due to the weak interbank ties. The key insight here is that the rescue costs are increasing in the level of the negative financial shock, and the potential losses are increasing in the level of the failing bank's prior interbank obligations. Therefore, "the ratio of the financial shock to the interbank obligations" become the key determining factor in banks' rescue decisions. As a result, if the integration is sufficiently low, then the failure of the shocked bank incurs sufficiently small costs to the rest of the system. In such a case, the remaining banks find the rescue of the shocked bank costly and prefer absorbing the losses that arise following the failure of the shocked bank instead of rescuing that bank. Correspondingly, non-rescue of the shocked bank can be eliminated via increasing the level of the potential losses to the system, so the interbank liabilities, up to a threshold level. For sufficiently high levels of interbank liabilities per bank, the remaining banks always find it profitable to rescue the shocked bank instead of letting it fail and absorbing the losses.

This result is related to the results in Acemoglu et al. [1] on how the shock level together with integration and diversification levels determine the extent of contagion. In their framework, for different shock and integration levels, complete network (high diversified) and ring network (low diversified) have different implications in terms of resilience and stability. Similarly, as shown in Elliott et al. [31], the diversification level play a key role if we consider only the contagion without rescue attempts. However, as shown in Theorem 1, whenever we consider the endogenously arising rescues, then the diversification level of each bank does not play key role in ultimate contagion as long as the distribution of links are such that the banks internalize the potential losses in the given network. Therefore, the overall network structure and the integration level of each bank as a function of the shock determine the ultimate stability of the system.

Theorem 1 part i is conditional on the fact that banks are adequately capitalized to

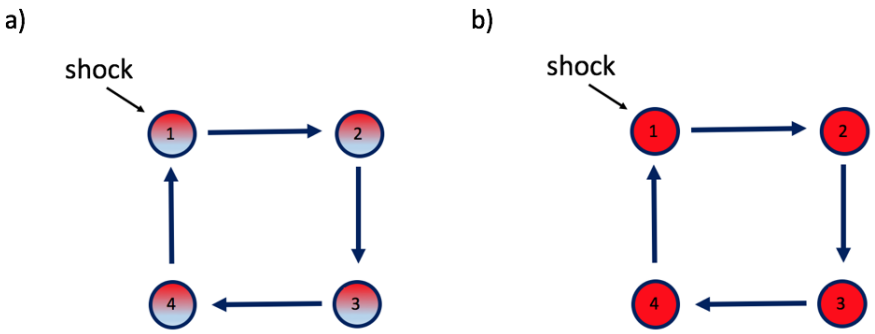

Figure 9: High integration causes non-preventable contagion for large shocks (Potential failures (panel a) and equilibrium (panel b) in Example 5)

rescue all distressed banks. Next, I consider what happens if the shock is sufficiently high such that banks do not have adequate aggregate capital (or capabilities) to prevent all potential failures.

4.2.2 High Levels of Interbank Exposures

Example 5 shows how high integration causes non-avoidable system-wide contagion whenever the shock is large. In Example 5, I consider a higher shock level and higher integration level than in previous examples, all else equal. Figure 9 depicts the potential failures and equilibrium outcome in Example 5, where banks are not able to form any rescue merger.

EXAMPLE **5** *Consider a financial network* (C, F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0.3 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \qquad \begin{array}{c} l_i = 0.93 \text{ for all } i \in N \\ p_i = 1 \text{ for all } i \in N \\ \sum_j C_{ji} = 0.3 \text{ for all } i \in N \\ s = 0.3 \text{ hits bank 1} \end{array}$$

Proposition 3 provides the related result.

PROPOSITION **3** *Consider a class of d-ring lattices* $\psi^R(\overline{c}, \overline{F})$ *such that* $N < N^*$ *and* $l > l^*$. *Then,*

i) there exists a set of mergers that can prevent all potential failures in any given $(\overline{c}, d, \overline{F}) \in \psi^R(\overline{c}, \overline{F})$ *iff the shock is sufficiently low,* $s \leq N(1-l)$,

ii) for sufficiently high levels of shock, $s > N(1-l)$, *there exists a set of mergers that can prevent "the potential contagious failures in full" in any given* $(\overline{c}, d, \overline{F}) \in \psi^R(\overline{c}, \overline{F})$ *iff the interbank liabilities per bank are sufficiently low,* $\overline{c} \leq (N-1)(1-l)$, *so that the potential losses from the initial failure are sufficiently low and the remaining banks are always able to absorb the losses from the failure of the shocked bank.*

As a result, even though high integration serves as an insurance mechanism whenever the shock is small, such benefits might reverse whenever the shock is large. I show that for sufficiently large shocks, banks can not rescue the bank hit by the shock. Nonetheless, it is always feasible to avoid potential contagious failures (if exist) via rescues as long as the interbank liabilities are sufficiently low. Consequently, the results so far imply that interbank liabilities entail a trade-off: strong ties among banks increase the rescue incentives, whereas they might cause a unavoidable systemic failure for large shocks. In Section 5.1, I analyze welfare-maximizing networks while considering both small and large shocks.

5 Welfare-Maximizing Interbank Networks and the Government Assistance

5.1 A Class of Welfare-Maximizing Interbank Networks without the Government Assistance in Rescues

Following the earlier results and examples, in this part, I provide results on network characteristics which would overcome inefficiencies in rescues. First, I define a class Ω of networks. The class Ω of networks is designed only by using the homogeneity in bank characteristics, which is relaxed in Section 5.3 and Section 5.4. In other words, there is no condition on the architecture of the network in the defined class Ω of financial networks except the condition in Assumption 1.[29] Formally, a financial network in the set of Ω has the following features:

$$\psi(C,\overline{F}) \in \Omega := \begin{cases} C_{ii} = 0 & \forall i \in N \\ C_{ij} \geq 0 & \forall i,j \in N \\ \sum\limits_{j \in N} C_{ji} \leq \frac{1}{2} & \forall i \in N \\ p_i = 1 & \forall i \in N \\ l_i = l < 1 & \forall i \in N \\ \zeta_i = 1 & \forall i \in N \\ N(1-l) \leq \frac{1}{2} & \end{cases}$$

Next, I redefine the shock environment. Different from the previous parts, there exists a shock s, which hits a randomly selected bank and can be either small or large with probabilities below. The large shock s_L represents the case that banks do not have adequate capital to avoid all potential failures.

[29] The only imposed condition on link formation is $\sum\limits_{j \in N} C_{ji} \leq \frac{1}{2}$. In addition, I consider that $N(1-l) \leq \frac{1}{2}$, which provides consistency in analysis under the condition $\sum\limits_{j \in N} C_{ji} \leq \frac{1}{2}$.

$$s := \begin{cases} s_S \in [0, N(1-l)] & \text{with probability } q \\ s_L \in (N(1-l), 1] & \text{with probability } 1-q \end{cases}$$

The sharing rule is the same as before. In the rest of the analysis, I focus on the welfare-maximizing strong Nash equilibrium.[30]

Given these specifications, the constrained optimization problem is as follows:

$$\max_{\psi(C,\overline{F})} W(\psi(C,\overline{F})^{M^*_{\psi(C,\overline{F})}}) \text{ subject to } \psi(C,\overline{F}) \in \Omega$$

where $M^*_{\psi(C,\overline{F})}$ is the equilibrium set of mergers that maximizes the social welfare in $\psi(C,\overline{F})$.

Next, I formally define *optimal* networks.

DEFINITION **12** *An interbank network $\psi^*(C,\overline{F}) \in \Omega$ is optimal if $W(\psi^*(C,\overline{F})^{M^*_{\psi^*(C,F)}}) \geq W(\psi(C,\overline{F})^{M^*_{\psi(C,\overline{F})}})$ for every interbank network $\psi(C,\overline{F}) \in \Omega$.*

Theorem 2 is the second main result of the paper.

THEOREM **2** *There exist c^* and c^{**} such that for $c^* \leq \overline{c} \leq c^{**}$ (integration is intermediate), any d-ring lattice $(\overline{c}, d, \overline{F}) \in \Omega$ which is potentially contagious such that $d < \frac{\overline{c}(1-\overline{c})}{1-l}$ or any d-ring lattice $(\overline{c}, d', \overline{F}) \in \Omega$ which is non-potentially contagious is an optimal interbank network.*

Theorem 2[31] provides a class of welfare-maximizing networks. First, non-clustering and sufficiently high interbank liabilities ensure that banks are incentivized to avoid all potential bankruptcies as long as feasible since any single bankruptcy poses sufficiently high costs to other banks. Second, sufficiently low interbank liabilities ensure that the remaining banks can always absorb the losses that would arise due to the initial failure. The absorption of losses due to the initial failure becomes crucial whenever the initial failure is unavoidable following a large shock.

In addition, the absorption of such losses might require rescue of the potential contagious failures in the rest of the system or not, which depends on the diversification level. Even though the diversification level plays key role in potential contagiousness, it does not play

[30] The existence of SNE and the welfare comparison among SNE for a given financial network $\psi(C,\overline{F}) \in \Omega$ is non-trivial. However, Theorem 2 shows that there exist networks in which SNE exists where the economy reaches the maximum possible level of social welfare for given bank characteristics. Therefore, the existence of SNE for any network $\psi(C,\overline{F}) \in \Omega$ has not been studied in this part.

[31] The result in Theorem 2 is given based on the potentially-contagiousness instead of the diversification level. The reason is that the dependency of creditors to a given bank might have small differences in decimals in the dependency matrix A, which would go to zero for sufficiently large N. These difference in decimals complicates the potential contagion ordering and the proof of Theorem 2 with no additional insight. In order to avoid these complications, I provide results for these cases separately.

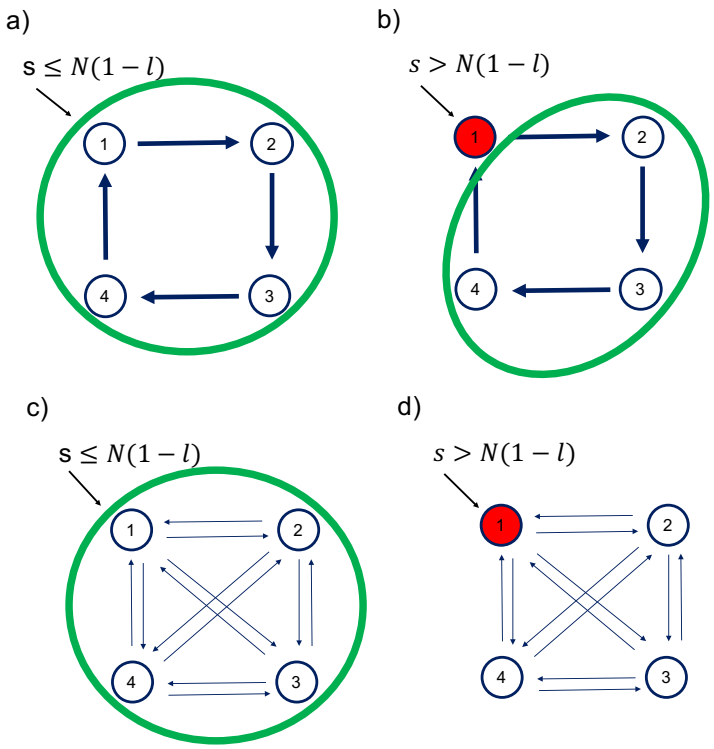

Figure 10: Examples of optimal networks (without government intervention)

key role in ultimate contagion because banks endogenously rescue each other and absorb the losses that would otherwise cause contagious failures.

Consequently, the result implies that potential contagiousness does not necessarily imply financial instability. Rather, when we consider the endogenously arising rescues, financial stability is determined by two main properties: *i)* potential bankruptcy losses internalized by banks, which depend on the overall structure of the interbank network, and *ii)* loss absorption capacity of the system (i.e., banks' aggregate capital), which depends on the primitives of the economy and is independent from the interbank network. For instance, the interbank network in Example 4, which is a potentially contagious network, is an example of an optimal interbank network. In addition, Figure 10 illustrates two optimal networks with different diversification levels. The ring network is potentially contagious, whereas the complete network is non-potentially contagious network. Bank characteristics and integration levels are same as given in Example 4.

The analysis so far has focused on rescue formation with no government assistance. The findings show that the government assistance is not required in certain networks. Next, I discuss welfare effects of government-assistance.

5.2 Government Assistance in Rescues

First, I revisit the timing of events. Before the rescue formation by banks, the government announces a transfer scheme $\vec{\mathbf{t}}$, which includes the information of the amount of the non-negative government assistance to each bank $i \in N$. The government assistance is realized after the shock hits and before the formation of rescues by banks. The vector of proprietary assets after the shock and the realization of the government assistance is given by $\mathbf{p} = [p_1 + t_1, ..., p_r - s + t_r, ..., p_n + t_n]'$.

Following the government assistance, banks form rescue consortia as previously defined in Section 3.

In other words, the government decides how much to assist in rescues while taking into consideration the potential contributions of banks in a perfect information framework.

The government transfers might come from a variety of sources such as tax collected from households and might be thought of as a cash injection that is designed to support the private-sector resolution of distressed banks.

The social welfare with the government intervention is redefined as follows:

DEFINITION **13** *The social welfare in* $(C, F)^M$ *with the government assistance is equal to* $W(C, F)^M = \sum_{i \in N^M} (p_i - b_i) - s - \eta \sum_{i \in N} t_i$ *where* $b_i = \beta_i$ *if* $i \in N^M$ *defaults, and* 0 *otherwise.* $\eta > 0$ *captures the relative inefficiency of government bailouts.*

Definition 13 captures the fact that the capital transfers by banks are more efficient than the bailout transfers by the government. This might be explained as follows. A bank's capital serves a safety net against its own failure risk. Accordingly, the minimum capital requirements are designed to ensure that each bank always holds sufficiently high levels of capital. In this (rescue) framework, the available capital in the system can be used for rescue transfers between banks and, hence, serves a safety net against any failure risk in the system. As a result, such capital transfers are transfers of resources that banks are obliged to hold. However, government bailouts are transfers to banks that would be otherwise used for providing funds to health care, education, or social services. To this end, I consider that the capital transfers between banks are costless, whereas the government bailouts incur some costs for the society. Consequently, the government bailouts are socially less efficient than the capital transfers between banks.

Parameter $\eta > 0$ captures the fact that the government bailouts are relatively less efficient than the capital transfers between banks (i.e. rescue mergers). For simplicity, I consider $\eta = 1$. Next, I define welfare-maximizing government and transfer scheme.

DEFINITION **14** *For any given financial network* (C, F), $\mathbf{t}$ *is a social welfare-maximizing government-assistance if* $\nexists \mathbf{t}'$ *such that*

$W(C, F, \mathbf{t}')^{M^*_{(C,F,\mathbf{t}')}} > W(C, F, \mathbf{t})^{M^*_{(C,F,\mathbf{t})}}$ *where* $M^*_{(C,F,\mathbf{t})}$ *and* $M^*_{(C,F,\mathbf{t}')}$ *are equilibrium set of mergers which maximize the social welfare in* $(C, F, \mathbf{t})$ *and* $(C, F, \mathbf{t}')$*, respectively. A government is social welfare-maximizer if it implements the social welfare-maximizing government-assistance for any given network* (C, F).

Correspondingly, I consider that the government is a social welfare-maximizer government. Proposition 4 shows the selective rescue decision of the government. When the shock is sufficiently large, the government prefers to assist in the rescue of the rest of the system instead of the rescue of the shocked bank. This result together with the previous results offers plausible explanations for the rescue decisions in the 2007-2009 period.

PROPOSITION **4** *Consider a potentially contagious d-ring lattice* $(\overline{c}, d, \overline{F})$ *such that* $d < \frac{\overline{c}(1-\overline{c})}{1-l}$ *, where there exists no rescue consortia that can "prevent all potential failures" or "prevent the potential contagious failures in full" without government-assistance. Then, there exists* s^* *such that*

i) for $s \leq s^*$*, all potential failures are prevented by a government-assisted rescue consortium,*

ii) for $s > s^*$*, the government does not assist in the rescue of the shocked bank, but the potential contagious failures are prevented in full by a government-assisted rescue consortium,*

where s^* *is increasing in the integration level* $\overline{c}$*.*

Proposition 5 revisits the class of network described in Section 5 and provides welfare-maximizing network structures with government intervention.

PROPOSITION **5** *Consider the class of optimal interbank networks provided in Theorem 2 (under no government intervention), where* $c^* \leq \overline{c} \leq c^{**}$ *holds for any such optimal interbank network* $(\overline{c}, d, \overline{F}) \in \Omega$*. Then, any such evenly-connected network* $(\overline{c}, d, \overline{F}) \in \Omega$ *where* $\overline{c} = c^{**}$ *is an optimal network with government intervention. The class of optimal networks with government intervention satisfies the following properties:*

i) for $s = s_s$*, all potential failures are avoided with no government assistance,*

ii) $s = s_L \leq \frac{[1+N(1-l)]}{2}$ *, all potential failures are avoided by a government-assisted rescue consortium,*

iii) for $s = s_L > \frac{[1+N(1-l)]}{2}$*, the government does not assist in the rescue of the shocked bank, but potential contagious failures (if exist) are avoided in full with no government-assistance.*

This last result shows that the optimal networks with government intervention are such that there is no need for the government assistance when the shock is small. On the other hand, whenever a large shock $s > N(1-l)$ hits, the government's decision about whether or not to assist in the rescue of the shocked bank depends on the shock level. Sufficiently high integration reduces the required amount of government transfer for rescuing the shocked bank and, hence, results in a welfare-improving government intervention. As a result, the highest threshold level of integration c^{**} together with evenly-connectedness emerge as a class of welfare-maximizing networks with government intervention. In such networks, banks' contributions in rescues are maximized unconditional on the shock level. Consequently, in such networks, the government assists in the rescue of the shock bank as long as the shock is smaller than the threshold level given in Proposition 5 part *ii*.

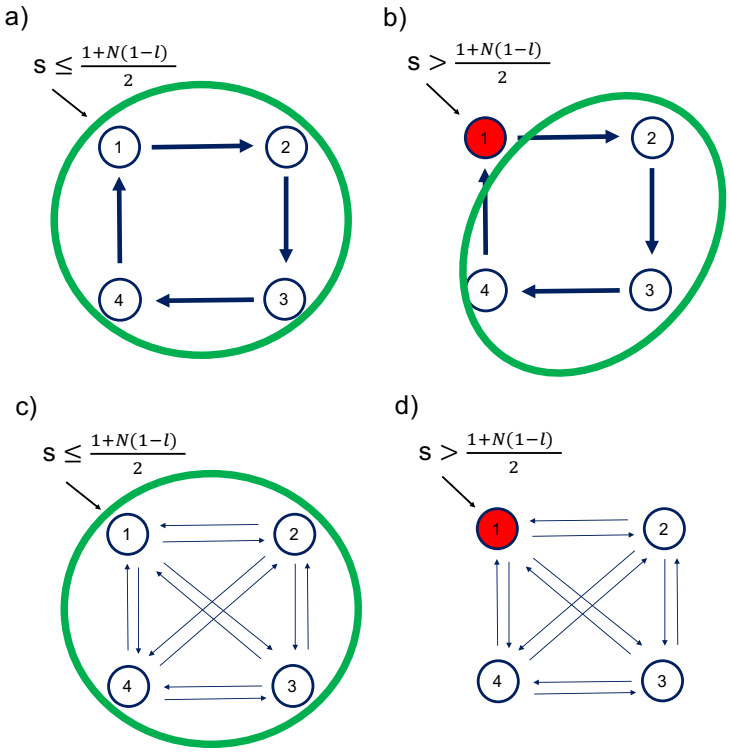

Figure 11: Examples of optimal networks with government intervention (Equilibrium for small shock (panel a) and large shock (panel b) in Example 6)

Example 6[32] and Figure 11 illustrate optimal networks with government intervention.

EXAMPLE **6** *Consider a financial network* (C, F) *with the following interbank claims and bank characteristics:*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0.21 \\ 0.21 & 0 & 0 & 0 \\ 0 & 0.21 & 0 & 0 \\ 0 & 0 & 0.21 & 0 \end{bmatrix} \qquad \begin{array}{c} l_i = 0.93 \text{ for all } i \in N \\ p_i = 1 \text{ for all } i \in N \\ \sum_j C_{ji} = 0.21 \text{ for all } i \in N \end{array}$$

The integration level in Example 6 is higher than the integration level in Example 4 that illustrates an optimal network with no government intervention. In addition, one of the other main differences between the two scenarios is that with the government intervention, rescues are still formed for relatively higher shocks compared to the case with no government intervention, as discussed above. For large shocks, banks would not be able to rescue the shocked bank without government assistance even if they have the incentives. Accordingly, for large shocks up to a threshold level, sufficiently high levels of government assistance makes the rescue of all potential failures feasible. This explains the welfare-improvement by the government intervention.

5.3 Rescues in Networks with Heterogenous Assets, Liabilities, and Interbank Exposures

The analysis in Section 5 so far has focused on a class of evenly-connected networks that provide us the key insight about rescue formation. In such evenly-connected networks, potential losses are internalized by banks in the system and, therefore, contributions in rescues are maximized. Next, I consider a class of non-evenly connected networks: star networks. The star structure can be described as follows. A single bank is located at the core and connected to all other banks and each other bank is connected to only the core bank. Next, I formally define star network and provide the result on star network.

DEFINITION **15** *A star network* (C, F) *that consists of a single core bank* $c \in N$ *and a set of periphery banks* $\{N \setminus c\}$ *is such that:*

[32] In Example 6, the optimal integration level $c^{**} = 0.21$ is rounded to the nearest 0.01.

$$(C,F) := \begin{cases} C_{cp} = \alpha_c & \forall p \in N \setminus c \\ C_{pc} = \alpha_p & \forall p \in N \setminus c \\ C_{pp'} = 0 & \forall p, p' \in N \setminus c \\ C_{cc} = C_{pp} = 0 & \forall p \in N \setminus c \\ p_c > p_p = 1 & \forall p \in N \setminus c \\ \sum\limits_{j \in N} C_{ji} \leq \frac{1}{2} & \forall i \in N \\ \sum\limits_{i \in N} (p_i - l_i) \leq \frac{1}{2} & \end{cases}$$

The integration level of the core bank and any periphery bank is equal to $(N-1)\alpha_p$ and α_c, respectively. I also normalize the proprietary asset value of any periphery bank to $p_p = 1$.

PROPOSITION **6** *Consider an economy with four banks. There exists exogenously given* (p_c, l_c, p_p, l_p) *and interbank claims* $\alpha_c^*(p_c, l_c, p_p, l_p)$ *and* $\alpha_p^*(p_c, l_c, p_p, l_p)$ *such that a star network with claims* $\alpha_c^*(p_c, l_c, p_p, l_p)$ *and* $\alpha_p^*(p_c, l_c, p_p, l_p)$ *is an optimal interbank network.*

Proposition 6 captures the heterogeneity in both asset sizes and connectivity. The result implies that for exogenously given proprietary assets and liabilities, a star network can be an optimal interbank network. For simplicity, I consider four banks in Proposition 6 that can be extended to n banks with the same insights. Next, I provide an example with four banks. As illustrated in Example 7, the integration level of periphery bank is higher than integration level of the core bank, which guarantees that the connectedness of each periphery bank to the rest of the system is at a sufficient level. As a result, the optimal level of integration and connectivity depends on the exogenously given asset sizes and external liabilities.

EXAMPLE **7** *For exogenously given* $p_p = 1$, $p_c = 5$, $l_p = \frac{42}{44} < 1$, *and* $l_c = \frac{1033}{220} < 5$ *and* $N = 4$, *the star network with* $\alpha_c^* = \frac{4}{10}$ *and* $\alpha_p^* = \frac{1}{10}$ *is an optimal network.*

In addition to the analysis on the star network, one might think about the core-periphery structure, which is another network structure that has been extensively studied in the literature. The core-periphery structure is in some sense an extended version of the star network, which includes some additional connectivity layers such as connectivity among core banks. Therefore, in addition to the insights driven from the star network, the optimal connectivity among core banks play a key role in the effectiveness of rescues in a core-periphery structure. The earlier results on evenly-connected networks provides us an idea about the connectivity of the core banks. Consequently, putting together the results about evenly-connected networks (for core-core connectivity) and star network (for core-periphery connectivity) guide us to understand optimal connectivity in a core-periphery network.

As a result, for exogenously given proprietary assets and external liabilities, the properties of welfare-maximizing networks depend on the primitives of the economy. Then, a

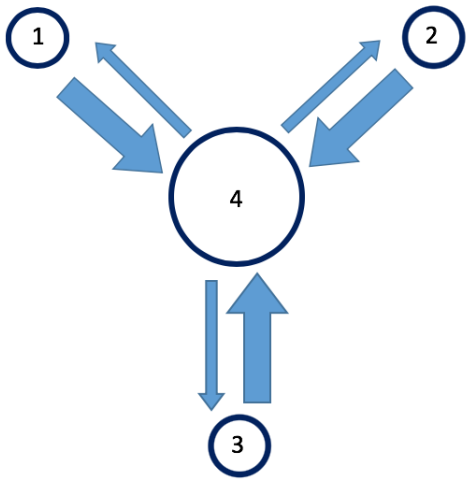

Figure 12: An optimal network structure in Example 7: *Star Network*

fundamental question arises: For exogenously given proprietary assets and external liabilities, what are the general properties of welfare-maximizing connectivity in the financial system? I answer this question in the next part.

5.4 Characterization of Welfare-Maximizing Interbank Exposures

In this part, I characterize the welfare-maximizing connectivity under no homogeneity constraint and with government assistance. For the constrained optimization, the class of networks Ω in this part is defined as follows:

$$\psi(C,\overline{F}) \in \Omega := \begin{cases} C_{ii} = 0 & \forall i \in N \\ C_{ij} \geq 0 & \forall i,j \in N \\ \sum\limits_{j \in N} C_{ji} \leq \frac{1}{2} & \forall i \in N \\ \sum\limits_{i \in N} (p_i - l_i) < \frac{1}{2} < min\{p_1, ..., p_n\} & \forall i \in N \\ v_i - l_i \geq 0 & \forall i \in N \end{cases}$$

The last two parts guarantees that for large shocks banks are not capable of rescuing all distressed banks and initially all banks are solvent.

THEOREM **3** *A financial network $(C,F) \in \Omega$ is a welfare-maximizing network iff (C,F) satisfies the following properties:*

- $(1 - A_{kk})\, p_k = \sum\limits_{j \in N \setminus k} (v_j - l_j)$ *for each singleton* $k \in N$ *(or equivalently* $\sum\limits_{i \in N \setminus k} \sum\limits_{j \in N \setminus k} A_{ij} p_j = \sum\limits_{j \in N \setminus k} l_j$*) (intermediate integration)*
- $\sum\limits_{i \in K} \sum\limits_{j \in K} A_{ij} p_j \leq \sum\limits_{i \in K} l_i$ *for all* $K \subset N$ *(no clustering of interbank exposures)*

The characterization result shows that there are two key properties of welfare-maximizing networks: interbank exposures that are formed with no clustering among any subset of banks, and intermediate level of interbank liabilities per bank. Part i implies that the total dependency to any given bank should be at intermediate level, which is similar to the result in previous parts. In addition, part ii implies that the network should be non-clustered such that for any given set of banks, the interbank exposures among any subset of banks should be smaller than a threshold level. The key insight is that i if bank i defaults (for large shocks), remaining banks always maintain stability with no government assistance, and ii banks contributions in rescue of the bank hit by the shock is always maximized due to the system-wide potential contagion risk. As a result, in such non-clustered connected networks, the available capital in the system is always used for injecting capital to the distressed banks whenever it is required, which minimize the government bailout transfers.

6 Conclusion

Recently, 2007-2008 global financial crisis has brought private-sector bank resolution methods to the forefront of the discussions and has shown that timely rescues play key role in maintaining stability in interconnected systems. On the other hand, there is a historical debate on bank rescues and financial stability going back to the discussions by Bagehot and Thornton in 19^{th} century. In this study, I develop a coalition formation framework to analyze bank failures and systemic risk while considering banks' rescue incentives against failure risks. The results provide general insights into how rescues can work as a self-correction mechanism in networks that face threats against individual or systemic stability. The results show how the system-wide structure of the interbank network play a key role in effectiveness of rescues by banks.

This study focuses on the counterparty risks while abstracting away from common asset exposures of banks. In order to fully understand the potential gains from private-sector resolution, the next step of research is investigating the endogenous formation of the interbank network and banks' asset portfolio in the pre-crisis period, which would help us understand whether the interbank obligations established in non-crisis period can work as a commitment mechanism for rescues in distressed times.

Lastly, although the focus of this study is financial networks, this study provides general insights into how agents behave collectively to resolve individual or systemic threats in networks where payoffs of agents are interdependent. As an example, one can consider the resolution of sovereign debt crises (e.g., European sovereign debt crisis), where different countries have different incentives to contribute to rescue plans. Another related example is the rescue of distressed firms in manufacturing sector (e.g., rescues in auto-industry in Japan and US), which are designed to avoid disruptions in supply chains. Analyzing rescues in a production economy requires different modeling and potentially incorporates different insights, which is left as a future research direction.

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7 Appendix

LEMMA **1** *i)* $A_{ij} = \sum_{k \in N} A_{ik} C_{kj}$ *for all* $i \neq j$.

ii) $A_{ii} = \hat{C}_{ii} + \sum_{k \in N} A_{ik} C_{ki}$ *for all* $i \in N$.

Proof of Lemma 1:

First, I show that $A = AC + \hat{C}$ holds for any given C matrix such that $\hat{C}$ is a diagonal matrix with entries $\hat{C}_{ii} = (1 - \sum_{j \in N} C_{ji}) > 0$ for all i and $\hat{C}_{ij} = 0$ for all $i \neq j$, where $A = \widehat{C}(I - C)^{-1}$.

Suppose that $A = AC + \hat{C}$ holds. Then,

$A = AC + \hat{C}$ implies

$\hat{C}(I - C)^{-1} = \hat{C}(I - C)^{-1}C + \hat{C}$, which can be rewritten as:

$\hat{C}(I - C)^{-1} = \hat{C}[(I - C)^{-1}C + I]$.

Since $\hat{C}$ is a diagonal matrix with non-zero diagonal elements,

$(I - C)^{-1} = [(I - C)^{-1}C + I]$ holds, which can be rewritten as:

$(I - C)^{-1}(I - C) = I$, and hence $I = I$ holds, which completes the proof.

$A = AC + \hat{C}$ implies that $A_{ii} = \hat{C}_{ii} + \sum_{k \in N} A_{ik} C_{ki}$ holds for all $i \in N$, and $A_{ij} = \sum_{k \in N} A_{ik} C_{kj}$ holds for all $i \neq j$.

end of proof.

LEMMA **2** *Consider a financial network* (C, F) *in which a set of banks* $\phi \subseteq N$ *form the set of mergers* $M = \{m_1, ..., m_n\}$*, which satisfies the properties below:*

$N^M = (N \setminus \phi) \cup M$

$p_j^M = p_j \qquad l_j^M = l_j \qquad \beta_j^M = \beta_j \qquad \forall j \in N \setminus \phi$

$p_{m_k}^M = \sum_{k \in m_k} p_k \qquad l_{m_k}^M = \sum_{k \in m_k} l_k \qquad \beta_{m_k}^M = \sum_{k \in m_k} \beta_k \qquad \forall m_k \in M$

Then, for any given set M*, there exist unique structures for* C^M *and* A^M *that satisfy*

$V_j^M = V_j \qquad L_j^M = L_j \qquad \forall j \in N \setminus \phi$

$V_{m_k}^M = \sum_{k \in m_k} V_k \qquad L_{m_k}^M = \sum_{k \in m_k} L_k \qquad \forall m_k \in M$

for every asset return vector $p \in \mathbb{R}_+^N$ *. The unique structures have the following properties:*

$C_{ij}^M = C_{ij} \ \forall i, j \in N \setminus \phi \qquad C_{m_k j}^M = \sum_{k \in m_k} C_{kj} \ \forall j \in N \setminus \phi$

$C_{m_k m_k}^M = \left(\frac{\sum_{k \in m_k} \sum_{l \in m_k} (C_{lk} V_k)}{\sum_{k \in m_k} V_k} \right) \forall m_k \in M \qquad C_{j m_k}^M = \left(\frac{\sum_{k \in m_k} (C_{jk} V_k)}{\sum_{k \in m_k} V_k} \right) \forall j \in N \setminus \phi$

$C_{m_k m_l}^M = \left(\frac{\sum_{l \in m_l} \sum_{k \in m_k} (C_{kl} V_l)}{\sum_{l \in m_l} V_l} \right) \forall m_k, m_l \in M$

$A_{ij}^M = A_{ij} \ \forall i, j \in N \setminus \phi \qquad A_{m_k j}^M = \sum_{k \in m_k} A_{kj} \ \forall j \in N \setminus \phi$

$A_{m_k m_k}^M = \left(\frac{\sum_{k \in m_k} \sum_{l \in m_k} (A_{lk} p_k)}{\sum_{k \in m_k} p_k} \right) \forall m_k \in M \qquad A_{j m_k}^M = \left(\frac{\sum_{k \in m_k} (A_{jk} p_k)}{\sum_{k \in m_k} p_k} \right) \forall j \in N \setminus \phi$

$A_{m_k m_l}^M = \left(\frac{\sum_{l \in m_l} \sum_{k \in m_k} (A_{kl} p_l)}{\sum_{l \in m_l} p_l} \right) \forall m_k, m_l \in M$

The restructuring of the claims can be summarized as follows:

i) the claims among the non-merged banks remain same,

ii) the claims that a given merger holds in a given non-merged bank is equal to the sum of the claims that each bank in that merger holds in that non-merged bank,

iii) the claims of a given non-merged bank in a merger is equal to the weighted sum of its claims in each bank in the merger,

iv) the claims of a given merger in another given merger is a weighted sum of the claims of each bank in the banks in the other merger.

Proof of Lemma 2:

The equations below are satisfied before the mergers:

$V_j = (\sum_{k\in\phi} C_{jk}V_k) + (\sum_{i\in N\setminus\phi} C_{ji}V_i) + p_j$

$\sum_{k\in m_k} V_k = (\sum_{k\in m_k}\sum_{j\in N} C_{kj}V_j) + \sum_{k\in m_k} p_k$

Call the claims matrix given in Lemma 2 C^*. The equations below are satisfied after the mergers:

$V_j^M = (\sum_{i\in N\setminus\phi} C^*_{ji}V_i^M) + \sum_{m_k\in M}(C^*_{jm_k}V^M_{m_k}) + p_j^M \;\forall j \in N\setminus\phi$

$V^M_{m_k} = (\sum_{j\in N\setminus\phi} C^*_{m_kj}V_j^M) + C^*_{m_km_k}V^M_{m_k} + (\sum_{m_l\neq m_k} C^*_{m_km_l}V^M_{m_l}) + p^M_{m_k} \;\forall m_k \in M$

I claim that C^* satisfies the given properties in Lemma 2. In order to show this, I rewrite the equations after the merger by using the asset returns and the new claims C^* after the merger, which are given in Definition 3:

$V_j^M = (\sum_{i\in N\setminus\phi} C_{ji}V_i^M) + \sum_{m_k\in M}\left(\frac{\sum_{k\in m_k}(C_{jk}V_k)}{\sum_{k\in m_k}V_k}V^M_{m_k}\right) + p_j \;\; \forall j \in N\setminus\phi$

$V^M_{m_k} = (\sum_{j\in N\setminus\phi}\sum_{k\in m_k} C_{kj}V_j^M) + \left(\frac{\sum_{k\in m_k}\sum_{l\in m_k}(C_{lk}V_k)}{\sum_{k\in m_k}V_k}\right)V^M_{m_k} + \sum_{m_l\neq m_k}\left(\frac{\sum_{l\in m_l}\sum_{k\in m_k}(C_{kl}V_l)}{\sum_{l\in m_l}V_l}\right)V^M_{m_l} + \sum_{k\in m_k} p_k$

We know that the financial system before the merger has a unique solution. We also know that the new system has a unique solution. Then, if we plug $V_j = V_j^M \;\forall j \in N\setminus\phi$ and $V^M_{m_k} = \sum_{k\in m_k} V_k \;\forall m_k \in M$ into the equations for V_j^M and $V^M_{m_k}$, we get:

$V_j = (\sum_{i\in N\setminus\phi} C_{ji}V_i) + \sum_{m_k\in M}(\sum_{k\in m_k} C_{jk}V_k) + p_j = (\sum_{k\in\phi} C_{jk}V_k) + (\sum_{i\in N\setminus\phi} C_{ji}V_i) + p_j$, and

$\sum_{k\in m_k} V_k = (\sum_{k\in m_k}\sum_{j\in N} C_{kj}V_j) + \sum_{k\in m_k} p_k$

These are the equations that we had before the mergers. Thus, from the uniqueness property, $V_j = V_j^M \;\forall j \in N\setminus\phi$ and $V^M_{m_k} = \sum_{k\in m_k} V_k \;\forall m_k \in M$ is also the unique solution for the system after the merger. So, C^* satisfies the properties given in Lemma 2.

Next, I show that A^* also satisfies the properties given in Lemma 2. The given properties in Definition 1 implies that $e_j = e_j^M \;\forall j \in N\setminus\phi$ and $e^M_m = \sum_{k\in m_k} e_k \;\forall m_k \in M$ holds for the shareholders' equity. Equation (7) implies $v = Ap$ holds before the shock, and hence

$e = v - l = Ap - l$ holds before the shock. Since l is given exogenously, $v = v^M$ must hold, and hence $A^M p^M = Ap$ must hold.

For any given (C, F), A is the unique Leontief inverse matrix derived from the given C matrix**.** By Definition 1, for a given set of mergers, one can find p^M and v^M. **N**ext, I show that $v^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$, which means that A^* satisfies the given properties in Lemma 2.

$v^M_j = \sum_{m_k \in M} (A^*_{jm_k} p^M_{m_k}) + \sum_{i \in N \setminus \phi} A^*_{ji} p^M_i = \sum_{m_k \in M} \left(\frac{\sum_{k \in m_k} (A_{jk} p_k)}{\sum_{k \in m_k} p_k} \right) \sum_{k \in m_k} p_k + \sum_{i \in N \setminus \phi} (A_{ji} p_i) = \sum_{i \in N} A_{ji} p_i = v_j$

holds for every $p \in \mathbb{R}^N_+$. Similarly,

$v^M_{m_k} = A^*_{m_k m_k} p^M_{m_k} + \sum_{m_l \neq m_k} (A^*_{m_k m_l} p^M_{m_l}) + \sum_{j \in N \setminus \phi} A^*_{m_k j} p^M_j = \left(\frac{\sum_{k \in m_k} \sum_{l \in m_k} (A_{lk} p_k)}{\sum_{k \in m_k} p_k} \right) \sum_{k \in m_k} p_k + \sum_{m_l \neq m_k} \left(\frac{\sum_{l \in m_l} \sum_{k \in m_k} (A_{kl} p_l)}{\sum_{l \in m_l} p_l} \right) \sum_{l \in m_l} p_l + \sum_{j \in N \setminus \phi} \sum_{k \in m_k} A_{kj} p_j = \sum_{k \in m_k} \sum_{j \in N} A_{kj} p_j = \sum_{k \in m_k} v_k$

holds for every $p \in \mathbb{R}^N_+$. Thus, $v^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$.

This completes the first part of the proof that C^M and A^M satisfy the given properties of mergers for every $p \in \mathbb{R}^N_+$.

Next, I show that there exist unique structures for C^M and A^M satisfying the desired properties for every asset return vector $p \in \mathbb{R}^N_+$.

First, I show that A^* is the unique structure satisfying the properties in Lemma 2 for every $p \in \mathbb{R}^N_+$.

I already showed that $v^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$ for any given M. Fix any given M. Suppose that there exist another matrix A^{**} satisfying the desired properties on total assets and liabilities for every $p \in \mathbb{R}^N_+$ for M. Then, $v^M = A^* p^M$ and $v^M = A^{**} p^M$ hold for every $p \in \mathbb{R}^N_+$, which implies $A^{**} p^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$. Then, $A^{**} \epsilon_j = A^* \epsilon_j$ $\forall e_j$ holds where ϵ_j is the vector such that j^{th} element of ϵ_j is equal to 1 and all other elements of ϵ_j are equal to ϵ.

$A^{**} \epsilon_j = A^* \epsilon_j \; \forall \, \epsilon_j$ implies $A^*_{kj} = A^{**}_{kj} \; \forall k, j$ as $\epsilon \to 0$. Then, this implies $A^* = A^{**}$. Thus, A^* is unique.

Next, I show the uniqueness of C^*. I already showed that $V^M = (I - C^*)^{-1} p^M$ holds for every $p \in \mathbb{R}^N_+$ for any given M. Suppose that there is another C^{**} satisfying $V^M = (I - C^{**})^{-1} p^M$ for every $p \in \mathbb{R}^N_+$ for M. Then, similarly, $(I - C^*)^{-1} = (I - C^{**})^{-1}$ must hold.

$(I - C^*)^{-1} = (I - C^{**})^{-1} \implies ((I - C^*)^{-1})^{-1} = ((I - C^{**})^{-1})^{-1} \implies I - C^* = I - C^{**}$ holds from the uniqueness of the inverse matrix, which also implies $C^* = C^{**}$. Thus, C^* is unique.

Lastly, $A^*$and C^* being the unique structures satisfying the desired properties for every $p \in \mathbb{R}^N_+$ for any given M implies that $A^* = \hat{C}^* (I - C^*)^{-1}$, where $\hat{C}^*$ is a diagonal matrix derived from C^*.

Thus, A^M and C^M are the unique structures satisfying the desired properties for every asset return vector for any given set of mergers, which completes the proof.

Next, I provide Definition 16, which I use extensively for the proofs.

DEFINITION **16** *Bank i's distress rank is equal to $\overline{dr}_i = t$ if bank i would default at step t of the contagion if there were no rescue attempt in financial network (C, F). For any healthy bank $i \in N \setminus \aleph_T$, bank i's distress rank is equal to $\overline{dr}_i = N$.* [33]

Proof of Proposition 1:

First, I provide the complementary result below. Proposition 7 is on the existence of rescue mergers in financial networks. Definition below follows from the contagion algorithm.

PROPOSITION **7** *Consider a potentially contagious financial network (C, F) and the financial network $(C, F)^M$ that is formed by the set of mergers M such that $\overline{dr}_i = \overline{dr}_j$ for all $(i, j) \in m_k$ for all $m_k \in M$, and $\nexists m_k \in M$ such that $r \in m_k$. Then, there exists no rescue merger $m_k \in M$.*

Proposition 7 highlights the fact that the existence of rescue mergers depends on the potential contagion structure. The result shows that a merger that only involves banks with same distress ranks does not prevent any failure unless there exists any other merger which is formed by banks with different distress ranks.

- Proof of Proposition 1 continued:

For $\zeta = 0$, $\aleph_T = \aleph_1$ (the contagion algorithm stops after one step) where the only potential failures are the potential first step failures. $1 - l - A_{ir}s < 0$ holds for all $i \in \aleph_1$, and $1 - l - A_{ir}s \geq 0$ holds for all $i \in N \setminus \aleph_1$.

By Proposition 7, there exists no rescue merger which is formed among banks in the set $\aleph_1 \setminus r$ or among banks in the set $N \setminus \aleph_1$.

Then, any rescue merger must involve at least one bank from each set $N \setminus \aleph_1$ and $\aleph_1 \setminus r$.

Consider an outcome of the game which involves at least one such rescue merger. Select any of these rescue mergers arbitrarily, and denote that merger by m_k. For any such outcome, the net worth of m_k is equal to $e^M_{m_k} = \sum_{i \in m_k \setminus \aleph_1} (v_i - l_i - A_{ir}s) + \sum_{i \in \{\aleph_1 \cap m_k\}} (v_i - l_i - A_{ir}s)$, where there exist $K - h$ number of healthy banks and h number of distressed banks involved in m_k.

For any distressed bank $i \in \{\aleph_1 \cap m_k\}$, $v_i - l_i - A_{ir}s < 0$ holds, and therefore, there always exists a healthy bank $i \in m_k$ that has a payoff less than $v_i - l_i - A_{ir}s$. Thus, given the strategies of other banks, revealing the strategy $s_i = \{i\}$ is a profitable deviation for any such healthy bank $i \in m_k$.

[33] The distress rank for a healthy bank can be at most N. In the extreme case, consider a network where only one bank defaults at each step of the algorithm and only one bank remains healthy ($N \setminus \aleph_T$ consists of a single bank). In that case, the distress rank is equal to $N - 1$ for the bank that fails lastly and, thus, the distress rank is equal to N for the remaining single healthy bank. This example shows that the distress rank is always less than or equal to N for a healthy bank in a given financial network, and that's why I fix it to N for generality.

This holds for any arbitrarily selected rescue merger m_k for any set of strategies resulting in set of merger M that involves at least one rescue merger.

Next, I show that there exists a SNE in which no merger has been formed. Consider the strategies such that $s_i = \{i\}$ for all $i \in N$, which results in an outcome $M = \emptyset$.

First, there exists no coalitional deviation among the banks in the set $N \setminus \aleph_1$ (healthy banks), since for any coalitional deviation the payoffs are summed. Similarly, for any coalitional deviation among the banks in the set $\aleph_1 \setminus r$, the payoffs would be equal to zero. In addition, as I showed above, any coalitional deviation that involves at least one healthy bank and one distressed bank from the set of $\aleph_1 \setminus r$ is a non-profitable deviation for any healthy bank involved in such a coalitional deviation. Lastly, for $l_r > l_r^*$, there exists no profitable coalitional deviation that involves bank r. Therefore, $s_i = \{i\}$ for all i is an equilibrium set of strategies.

Proof of Proposition 2:

By Lemma 1, $A_{ki} = 0$ for any $k \in N \setminus R$, $i \in R$ where R is the set of banks located in the connected component that includes bank r. This implies that any $k \notin R$ is a healthy bank and has a net worth of $e_j = 1 - l - A_{ir} = 1 - l$.

Next, consider the set of mergers M as an outcome of the game. Consider that there exists a merger $m_k \in M$ which includes a healthy bank h. For m_k to be a rescue merger, there must be a bank $i : i \in m_k \cap R$. Consider the set of mergers $M' = M \setminus m_k$, an alternative outcome of the game. $e_h^{M'} = 1 - l$ holds for any healthy bank $h \in m_k$. Then, if $e_h^M < 1 - l$ holds for any healthy bank $h \in m_k$, then the strategy $s'_h = \{h\}$ is always a profitable deviation for h, and M cannot be an equilibrium set of mergers in that case. So, for M to be an equilibrium outcome, $e_h^M \geq 1 - l$ must hold, and thus $e_{m_k}^M \geq 1 - l$ must also hold.

Suppose that $e_{m_k}^M \geq 1 - l$. Then, either there exists at least one more healthy bank in m_k, or there exists at least one distressed bank j in m_k such that j is a healthy bank in $(C, F)^{M \setminus m_k}$. Otherwise, if all banks in $m_k \setminus h$ are distressed banks in $(C, F)^{M \setminus m_k}$, then $v_j^{M \setminus m_k} < l_j$ holds for all $\{j : j \neq h, j \in m_k\}$, which implies that $e_{m_k}^M < 1 - l$. As a result, M cannot be an equilibrium outcome.

Consider that h is the only healthy bank in m_k, but there exists at least one distressed bank j such that j would not default in $(C, F)^{M \setminus m_k}$. Denote the set of such banks $\{j : j \in m_k \cap (N^{M \setminus m_k} \setminus \aleph^{M \setminus m_k})\}$ by D. Then, for any arbitrarily selected sharing rule, e_j^M holds if $e_j^M < e_j^{M \setminus m_k} = 1 - l$ holds for some $j \in D$. In this case, $s'_j = \{j\}$ is a profitable deviation for any $\{j : j \in m_k \cap (N^{M \setminus m_k} \setminus \aleph^{M \setminus m_k})\}$, and hence M cannot be an equilibrium outcome.

Next, consider that there exists more than one healthy bank in m_k. In this case, for $e_j^M \geq 1 - l$ to hold, either all banks are healthy banks and reallocate the net worth, which implies that m_k is never a rescue merger, or if m_k is a rescue merger the condition above holds for some $j \in D$ or there exists a healthy bank $i \in m_k$ s.t. $e_i^M < 1 - l$. Then, for any such bank i having $e_i^M < 1 - l$, $s'_i = \{i\}$ is a profitable deviation. As a result, M cannot be an equilibrium outcome in this case as well. This completes the proof by showing that there

exists no equilibrium outcome in which any bank in $N \setminus R$ is involved in a rescue merger. Next, by Proposition 3, for $s > (d+1)(1-l)$ and $c > c^*(s,l,d)$, there exists no rescue merger which is formed among the banks in the set R.

Lastly, similar argument in the proof of Proposition 1 follows here, and $s_i = \{i\}$ for all i is an equilibrium set of strategies.

Proof of Proposition 3:

Proof of part i)

For $s > N(1-l)$, if there exists no distressed bank, then $v_i - l - A_{ir}s \geq 0$ must hold. If this holds for all i, then $\sum_i (v_i - l - A_{ir}s) = N(1-l) - s \geq 0$ holds as well. However, for $s > N(1-l)$, it is a contradiction. Thus, for $s > N(1-l)$, there exists at least one distressed bank.

By Lemma 1 and Assumption 1, $A_{ii} = (1-c) + \sum_{k \in N} A_{ik}C_{ki} > \frac{1}{2}$, and since A matrix is a column-stochastic matrix, in any given d-ring lattice (c,d,F), $A_{ii} > \max_{j \neq i}\{A_{ij}\}$ holds for any $i \in N$. Thus, bank r is a distressed bank in this case. Then, r must be included in a merger to prevent all potential failures.

If there exists no other $i \in \aleph_1$, then the grand coalition of all banks, denoted by G, is the most resilient merger that involves bank r. To see that, first observe that $v_i - l - A_{ir}s > 0$ holds for all $i \neq r$ in this case.

Excluding a set T of banks from the grand coalition decrease the exposure of the remaining banks in the coalition G/T to the shock by $\sum_{t \in T} A^M_{tr}$. However, it also decrease the payoff of the coalition by $\sum_{t \in T}(v_t - l_t)$. For having a set of mergers M, which can prevent all potential failures, any configuration among the banks in T must remain healthy and the coalition G/T must remain healthy as well. Consider that this is true, then, since $v_i - l - A_{ir}s > 0$ holds for all $i \neq r$, excluding such set of banks decrease the resilience of the coalition that involves bank r even though the excluded set of banks remain solvent. This holds for any $T \subseteq N \setminus r$.

However, for $s > N(1-l)$, the payoff of grand coalition is equal to $N(1-l) - sA_{GG} = N(1-l) - s$, which follows from Lemma 2. Therefore, there exists no mergers which can prevent all failures.

On the other hand, if there exists some other $i \in \aleph_1$, then each bank $i \in \aleph_1$ must be involved in a merger. A similar argument applies in this case as well. The grand coalition is the most resilient merger that involves all banks in the set $\aleph_1$, but it defaults if formed. Thus, there exists no set of mergers which can prevent all potential failures. On the other hand, for $s \leq N(1-l)$, the grand coalition can prevent all potential failures.

Proof of part ii)

For $s \leq N(1-l)$, I already showed that the grand coalition of all banks can prevent all potential failures, and hence all potentially contagious failures can be prevented.

For $s > N(1-l)$ there exists no such merger. In that case, for $r \notin \phi$, in order to prevent potential contagious failures in full (if exists), all banks in $\aleph_2 \setminus r$ must be involved in a merger.

Otherwise, any bank in $\aleph_2 \setminus r$ would default since $v_i - l_i - A_{ir}(s + (1-s)\zeta) < 0$ holds for all $i \in \aleph_2$.

Similar to the proof in part i, if the coalition of banks $N \setminus r$ cannot prevent potential contagious failures in full, then no other coalition excluding bank r can prevent the potential contagious failures in full. This follows from the fact that:

i) each bank in $\aleph_2$ must be involved in a merger to prevent potential contagious failures in full,

ii) $v_i - l_i - A_{ir}s \geq 0$ holds for all $i \notin \aleph_2$.

In that case, the payoff of coalition of banks $N \setminus r$ is equal to $(N-1)(1-l) - \sum_{i \in N \setminus r} A_{ir}(s + (1-s)\zeta)$. Therefore, for $(N-1)(1-l) > \sum_{i \in N \setminus r} A_{ir}$, then there exists a set of mergers that can prevent potential contagious failures in full.

By the column stochasticity of the A matrix, the default of bank r reduces the remaining banks' total net worth by $(1 - A_{rr})$. In addition, Lemma 2 implies that $A_{Gr} = 1 - A_{rr}$ holds, where G is the coalition of banks $N \setminus r$. Therefore, $(N-1)(1-l) - A_{Gr} = (N-1)(1-l) - (1 - A_{rr}) \geq 0$ must hold.

Lemma 1 implies that for a finite N, $A_{ii} > 1 - \overline{c}$ and $1 - A_{ii} < \overline{c}$ holds for all $i \in N$ and for all $1 \leq d \leq N-1$.

Therefore, for $\overline{c} \leq \frac{(N-1)(1-l)}{s+(1-s)\zeta}$, $(N-1)(1-l) - \sum_{i \in N \setminus r} A_{ir}(s + (1-s)\zeta) > 0$ holds, and hence, there exists a set of mergers that can prevent potential contagious failures in full regardless of the diversification level. There exist N^* and l^* such that for $N < N^*$ and $l > l^*$, and for $\zeta = 1$, there exists such $\overline{c} \leq (N-1)(1-l)$.

Proof of Theorem 1:

Step 1) For $\zeta = 1$, first I show that in a potentially contagious d-ring lattice $(\overline{c}, d, \overline{F})$ such that $1 - l < \frac{\overline{c}(1-\overline{c})}{d}$, all banks are distressed banks.

Under Assumption 2, In any $d-$ring lattice if there exists any bank i with $\overline{dr}_i = 1$, then $\overline{dr}_r = 1$ holds.

The C matrix of a given d-ring lattice $(\overline{c}, d, \overline{F})$ is a circulant matrix, where each row vector is rotated one element to the right relative to the preceding row vector.

By Theorem 3.2.4 of Davis [22], if A and B are circulant matrices of order n and α_k scalars, then $\alpha_1 A + \alpha_2 B$ and AB are also circulant matrices. This theorem implies that $I - C$ is also a circulant matrix, since I is a circulant matrix as well. Theorem 3.2.4 of Davis [22] shows that if A is a non-singular circulant matrix, then A^{-1} is also a circulant matrix. Given that $(I-C)$ is invertible, $(I-C)^{-1}$ is also a circulant matrix. Finally, $A = \hat{C}(I-C)^{-1}$ is also a circulant matrix since $\hat{C}$ and $(I-C)^{-1}$ are circulant matrices of order n. This implies that $\{A_{ii}\}^d = \{A_{jj}\}^d$ holds for any given $d$where $\{A_{ii}\}^d$ is the A_{ii} of bank i in (c, d, f) with diversification level d. Moreover, $A_{i,i+k} = A_{j,j+k}$ for all $i, j \in N$ for any $k \leq N-1$ where $i+k$ is the k^{th} immediate neighbor of node i.

Therefore, for $1 - l < \frac{\overline{c}(1-\overline{c})}{d}$, by Lemma 2, for any $j \in d_r^{out}$, $A_{jr} > \frac{\overline{c}(1-\overline{c})}{d}$, and each bank $j \in d_r^{out}$ is a distressed bank.

Given that the A matrix is circulant and $\zeta = 1$, the losses of any single bank from $r+d+1$

to $r+2d$ is greater than $\frac{\overline{c}(1-\overline{c})}{d}$, which implies that any single bank from $r+d+1$ to $r+2d$ is also a distressed bank. The contagion algorithm proceeds similarly, and thus all banks are distressed banks.

Step 2) Consider an outcome of the game such that $M = \{N \setminus r\}$. Next, I show that there exists no profitable coalitional deviation by any set T of banks such that $T \subsetneq N \setminus r$.

The net worths in $(\overline{c}, d, \overline{F})^M$ are such that $e_r^M = 0$ and $e_{N\setminus r}^M = (N-1)(1-l) - (1-A_{rr})$.

Consider any coalitional deviation by set $T \subsetneq N \setminus r$, where T excludes some members of set d_r^{out}. Denote the set of such excluded banks by E, where $\{d_r^{out} \setminus E\} \subseteq T$. Following such a coalitional deviation, any single bank in set E defaults since $r \notin T$ and any single bank in E is a creditor of the shocked bank. Next, I show that such an outcome cannot be an equilibrium.

Rank the members in E based on their distance to r where i^{th} member of E is denoted by $r+i$, where $i \leq d$.

First, consider that the set E is a singleton. Suppose that the coalition T involves $(r+i+i)^{th}$ node. Then, whenever coalition T is a strictly profitable deviation, then the coalition $T \cup i$ is also a strictly profitable deviation. To see that, involving the $(r+i)^{th}$ bank into the coalition T would increase the total payoff of T by $1-l-A_{r+i,r}$ and would eliminate the loss of coalition T due to the failure of bank $r+i$, which is greater than $A_{r+i+i,r+i}$. Given that the A matrix of a d-ring lattice is a circulant matrix, $A_{r+i+i,r+i} = A_{r+i,r}$. As a result, whenever coalition T is a strictly profitable deviation, then the coalition $T \cup i$ is also a strictly profitable deviation.

However, coalition $T \cup i$ is not a strictly profitable deviation since involving any subset of banks in $N \setminus \{T \cup i\}$ to the coalition $T \cup i$ increase the individual payoffs of banks in $\{T \cup i\} \setminus \aleph_2$. Therefore, any profitable coalitional deviation T cannot involve the $(r+i+i)^{th}$ bank. If $(r+i+i)^{th}$ bank is not involved in T, then it defaults, and the same argument applies for $(r+i+i+i)^{th}$ bank. Thus, in any strictly profitable deviation, $(r+j\times i)^{th}$ nodes must be excluded.

Then, given that $i < d$, all $(r+j\times i)^{th}$ nodes default if they are excluded, and thus the loss of such T is higher than $\frac{\overline{c}(1-\overline{c})}{d} \mid \{T \setminus d_r^{out}\} \mid$. Moreover, any $k \in d_r^{out} \cap T$ has losses higher than $\frac{\overline{c}(1-\overline{c})}{d}$. Combining these together, any such T that excludes all $(r+j\times i)^{th}$ nodes would default.

If E is not a singleton (more than one creditor of r is excluded from the rescue consortia), then the same results hold and any such coalitional deviation would also default.

Step 3) Following Step 2 of the proof, next consider any coalitional deviation that includes bank r. Similar to the previous proofs, if the deviation of grand coalition is not profitable, then any coalitional deviation by $T \cup r$ such that $T \subsetneq N \setminus r$ cannot be a profitable deviation as well.

Next, consider the payoffs when the grand coalition deviates and rescues the shocked bank.

The total payoff would be equal to $N(1-l) - s$. On the other hand, if there is no such

deviation, then the total payoff of $N \setminus r$ is equal to $(N-1)(1-l)-(1-A_{rr})$.

Then, there exists no profitable deviation if $(N-1)(1-l)-(1-A_{rr}) \geq N(1-l)-s$.

By Lemma 1, $(1-A_{rr}) < \overline{c}$. Therefore, for $\overline{c} < s-(1-l)$, there exists no such coalitional deviation.

Proof of part i follows from the the previous steps, and for $\overline{c} \geq s-(1-l)$, there exists a SNE in which banks form the grand coalition and rescue the shocked bank. Otherwise, $N \setminus r$ is always a profitable coalitional deviation.

Proof of Theorem 2:

If a network satisfies the following properties, then it is an optimal network:

i) for $s = s_S$, there exists a strong Nash equilibrium in which all potential failures (if exist) are prevented.

ii) for $s = s_L$, there exists a strong Nash equilibrium in which potential contagious failures are prevented in full (if exist).

Any network satisfying the properties above weakly dominates any other network because the social welfare in such a network is equal to $W(c,d,F)^M = q[(\sum_{i \in N^M} p_i) - s] + (1-q)(\sum_{i \in N^M \setminus r} p_i)$, which is the maximum possible value for $W(C,F)^M$ in any given network for the given shock and external liabilities for the given set of banks.

The sufficient condition for capability of rescuing all potentially contagious failures (if exist):

$$[(N-1)(1-l)-(1-A_{ii}(C,F))] \geq 0 \tag{8}$$

Case 1) Potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$:

In any potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$ such that $1-l < \frac{\overline{c}(1-\overline{c})}{d}$, then the condition for $\overline{c}$ in Theorem 1 applies here as well. Thus, for $\overline{c} \geq s_s-(1-l)$, there exists a SNE in which banks form the grand coalition and rescue the shocked bank in any potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$ such that $1-l < \frac{\overline{c}(1-\overline{c})}{d}$.

For any given small shock $s_s \leq N(1-l)$, there exists a $\frac{1}{2} \geq \overline{c} \geq s_s-(1-l)$ for $(N-1)(1-l) \leq \frac{1}{2}$.

So, there always exists a SNE in which the first failure is prevented in a contagious d-ring lattice $(\overline{c},d,\overline{F})$ such that $d < \frac{\overline{c}(1-\overline{c})}{1-l}$.

Next, consider the large shock, s_L.

In any given network $\psi(C,F) \in \Omega$, following a large shock, there exists at least one distressed bank and there exists no set of mergers which can prevent all potential failures. Then, the desired condition in this case is given by Equation (8). For $(N-1)(1-l) \geq \overline{c}$, $(N-1)(1-l) \geq \overline{c} > (1-A_{rr}(\overline{c},d,\overline{F}))$ also holds and banks can prevent all potential contagious failures in a given potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$.

Combining these two conditions together, for $s_s-(1-l) \leq \overline{c} \leq (N-1)(1-l)$, any potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$ such that $d < \frac{\overline{c}(1-\overline{c})}{1-l}$ is an optimal network.

Case 2) Non-potentially contagious d-ring lattice $(\overline{c},d,\overline{F})$:

In any non-potentially contagious network, equation (8) strictly holds.

Therefore, in any non-potentially contagious $(\overline{c}, d, \overline{F})$, given the sharing rule, whenever equation (9) below holds, then such a network is optimal.

$$N(1-l) - s_S \geq (N-1)(1-l) - (1 - A_{ii}(C, F)) \tag{9}$$

Thus, for $\overline{c} \geq s_s - (1-l)$, there exists a SNE in a non-potentially contagious $(\overline{c}, d, \overline{F})$ in which banks form the grand coalition and rescue the shocked bank. Therefore, a non-potentially contagious d-ring lattice $(\overline{c}, d, \overline{F})$ such that $s_s - (1-l) \leq \overline{c} \leq (N-1)(1-l)$ is also an optimal network.

The set of non-potentially contagious d-ring lattices satisfying the condition $s_s - (1-l) \leq \overline{c} \leq (N-1)(1-l)$ depends on the level of s_S. For instance, for $s_S = N(1-l)$, the only non-potentially contagious network would be the complete network, and for smaller values for s_S, the set extends.

Lastly, as shown in proof in Theorem 1, since the A matrix of a d-ring lattice is a circulant matrix, $A_{ii} = A_{jj}$ for all i, j holds, thus the result holds for arbitrarily selected shocked bank r, and hence the result holds under the condition that the shock hits a single bank uniformly at random.

Proof of Proposition 4:

For such a network $(\overline{c}, d, \overline{F})$, $[N(1-l) - s] < 0$ and $[(N-1)(1-l) - (1 - A_{rr}(\overline{c}, d, \overline{F}))] < 0$. The minimum amount of the required government transfer to prevent all potential failures is equal to $s - N(1-l)$. The social welfare in this case would be equal to $W(\overline{c}, d, \overline{F})^{M^*_{\psi(\overline{c},d,\overline{F})}} = (\sum_{i \in N^M} p_i) - s - (s - N(1-l))$, if the grand coalition is formed with the minimum amount of the required government transfer and all potential failures are prevented.

On the other hand, if the government does not assist the rescue of the shocked bank, but assists in the rescue of the contagious failures in full, then the minimum amount of the required government transfer is equal to $[(1 - A_{rr}(\overline{c}, d, \overline{F})) - (N-1)(1-l)]$. In that case, the social welfare is equal to $W(c, d, F)^{M^*_{\psi(\overline{c},d,\overline{F})}} = (\sum_{i \in N^M} p_i) - s - [(1 - A_{rr}(\overline{c}, d, \overline{F})) - (N-1)(1-l)]$, if the grand coalition excluding bank r is formed with the minimum amount of the required government transfer, and the contagious failures are prevented in full.

On the other hand, Theorem 1 implies that the required government transfer for preventing the contagious failures partially would be higher than the required government transfer for preventing the contagious failures in full. To see that, the government always prefers to assist in the rescue of any potential contagious failure since the bankruptcy cost of such a bank is always greater than the required amount of transfer. So, a social welfare-maximizer government leaves no contagious failures out. Lastly, the required government transfer to prevent all potential contagious failures is minimized whenever the grand coalition or the coalition of $N \setminus r$ is formed, which follows from Theorem 1.

Next, I compare the social welfare under two cases: i) $N^M = \{N\}$, and ii) $N^{M'} = \{r, (N \setminus r)\}$.

For $(\sum_{i\in N^M} p_i) - s - [(1 - A_{rr}(\bar{c}, d, \overline{F})) - (N-1)(1-l)] \leq (\sum_{i\in N^M} p_i) - s - (s - N(1-l))$, the government prefers to assist in the rescue of the shocked bank. Given that with no intervention all banks would fail and the shareholders payoff would be equal to zero, then there exists no coalitional deviation as shown in the proof of Proposition 1, and the shocked bank would be rescued at equilibrium, where the promised government transfer is $(s - N(1-l))$ for the grand coalition and zero for any other merger configuration.

The condition for the shock can be rewritten as $s \leq s^* = [(1 - A_{rr}(\bar{c}, d, \overline{F})) + (1-l)]$. For given vector of external liabilities $\mathbf{l}$, s^* is increasing in $(1 - A_{rr}(\bar{c}, d, \overline{F}))$, which is increasing in the integration level $\bar{c}$, by Lemma 2 in Elliott et al. (2014).

For, $s > s^* = [(1 - A_{rr}(\bar{c}, d, \overline{F}) + (1-l)]$, the government prefers to assist in the rescue of the rest of the system, and there exists an equilibrium in which all potential contagious failures are prevented, where the promised government transfer is $(1 - A_{rr}(\bar{c}, d, \overline{F})) - (N-1)(1-l)$ for the coalition $(N \setminus r)$ and zero for any other merger.

Proof of Proposition 5:

Theorem 2 implies that for $s = s_S$, all potential failures are prevented with no government assistance in such a network. For $s = s_L$, the government assistance is required to prevent all potential failures.

Next, I show that there exists $\overline{s} = \frac{[1+N(1-l)]}{2}$ such that the government never assists in the rescue of all potential failures if the large shock $s_L > \overline{s} = \frac{[1+N(1-l)]}{2}$.

For any level of large shock $s_L > N(1-l)$, the minimum level of the required amount of the government transfer is equal to $t_r = s_L - N(1-l)$. Therefore, the government never assists in the rescue of the shocked bank in any network $(C, F) \in \Omega$ if $N - s_L - [s_L - N(1-l)] < N-1$, which can be rewritten as $s_L > \frac{[1+N(1-l)]}{2}$.

For $s = s_L$, the social welfare with no government assistance in an optimal d-ring lattice is equal to $W(\bar{c}, d, \overline{F})^{M^*_{\psi(\bar{c},d,\overline{F})}} = (\sum_{i\in N^M} p_i) - p_r = N-1$. On the other hand, if the government transfers sufficiently high amount of resources to the shocked bank so that the "rescue of all potential failures" emerges as an equilibrium outcome, then the social welfare is equal to $W(c, d, F)^{M^{**}_{\psi(\bar{c},d,\overline{F})}} = (\sum_{i\in N^M} p_i) - s_L - t$ where t is the sum of the government-assistance. Thus, the government assists in the rescue of all potential failures if $(\sum_{i\in N^M} p_i) - s_L - t \geq (\sum_{i\in N^M} p_i) - p_r$, which can be rewritten as $N - s_L - t \geq N - 1$.

Proposition 4 implies that the transfer of $t_r = s_L - N(1-l)$ results in the prevention of all potential failures in a d-ring lattice $(\bar{c}, d, \overline{F})$ such that $(N-1)(1-l) - (1 - A_{rr}(\bar{c}, d, \overline{F})) = 0$.

Thus, in such a d-ring lattice, whenever $N - s_L - (s_L - N(1-l)) \geq N - 1$, the government always prefers to assist in the rescue of all potential failures. The condition can be rewritten as $s_L \leq \frac{[1+N(1-l)]}{2}$. Moreover, the required amount of transfer is equal to the possible minimum amount of transfer that is required to prevent all potential failures in any network $(C, F) \in \Omega$, which is equal to $s_L - N(1-l)$. Lastly, since the government is a social welfare-maximizer, the social welfare is equal to $N - s_L - [s_L - N(1-l)] \geq N - 1$ for

$s_L \leq \frac{[1+N(1-l)]}{2}$, and the social welfare is equal to $N-1$ for $s_L > \frac{[1+N(1-l)]}{2}$.

The integration level $\overline{c}$ that satisfies $(N-1)(1-l) = (1 - A_{rr}(\overline{c}, d, \overline{F}))$ is an intermediate integration since $(N-1)(1-l) = (1 - A_{rr}(\overline{c}, d, \overline{F})) > \overline{c}(1-\overline{c})$, where $\overline{c}(1-\overline{c})$ is increasing in $\overline{c}$ for $\overline{c} \in [0, \frac{1}{2}]$. This completes the proof.

Proof of Proposition 7:

Index the mergers in M based on the distress ranks of the banks involved, and denote a given merger by m_k if $\overline{dr}_i = \overline{dr}_j = k \; \forall (i,j) \in m_k$, and denote the set of mergers m_k by M_k. Consider that bank r is not involved in any merger, $r \notin \phi$.

Suppose that there exists a merger $m_1 \in M$. We know that $v_i - l_i - A_{ir}s < 0$ holds for all $i \in m_1$. By Lemma 2, $A^M_{m_1 r} = \sum_{i \in m_1} A_{ir}$ holds since $r \notin \phi$. Then, $\sum_{i \in m_1}(v_i - l_i) - A^M_{m_1 r} = \sum_{i \in m_1}(v_i - l_i) - \sum_{i \in m_1} A_{ir} < 0$ also holds, which means that m_1 defaults at step 1 in $(C,F)^M$. This result holds for any $m_1 \in M_1$. On the other hand, by Lemma 2, $A_{jr} = A^M_{jr} \; \forall j \notin \phi$, which implies that $v_j - l_j - A_{jr} = v_j - l_j - A^M_{jr} < 0$ holds for all $\{j : j \notin \phi, \overline{dr}_j = 1\}$; thus any such bank j defaults at step 1 in $(C,F)^M$. Lastly, Lemma 2 implies that any $\{j : j \notin \phi, \overline{dr}_j > 1\}$ remains solvent at step 1 in $(C,F)^M$. Similar argument follows for each subsequent step of the algorithm. Formally, for $p_i = 1$ for all $i \in N$, and $\beta_i = \overline{\beta}$ for all $i \neq r$, we have:

$A^M_{m_k m_l} \sum_{l \in m_l} \beta_l = \left(\frac{\sum_{l \in m_l} \sum_{k \in m_k} (A_{kl} p_l)}{\sum_{l \in m_l} p_l} \right) \sum_{l \in m_l} \beta_l = (\sum_{l \in m_l} \sum_{k \in m_k} A_{kl}) \overline{\beta}$.

Therefore, for any m_k, its total loss due to the failures up to the k^{th} step of the contagion algorithm in $(C,F)^M$ is equal to $(\sum_{l:l<k, l \in M_l} \sum_{k \in m_k} A_{kl})\overline{\beta} + \sum_{i \notin \phi, i \in \aleph_{k-1}} \sum_{k \in m_k} A_{ki})\overline{\beta} + \sum_{k \in m_k} A_{kr}(s + \beta_r)$, which is equal to the sum of the individual losses of each bank in m_k due to the failures up to the k^{th} step of the contagion algorithm that would be realized if merger m_k has not been formed, all else equal.

For any individual bank that has the distressed level of k, we know that $v_k - l_k - A_{kr}(s + \beta_r) - (\sum_{i \in \aleph_{k-1}} A_{ki})\overline{\beta} < 0$ holds. Thus, $\sum_{k \in m_k}(v_k - l_k - A_{kr}(s+\beta_r)) - (\sum_{l:l<k, l \in M_l} \sum_{k \in m_k} A_{kl})\overline{\beta} - (\sum_{i \notin \phi, i \in \aleph_{k-1}} \sum_{k \in m_k} A_{ki})\overline{\beta} < 0$ also holds, and hence; any m_k defaults at step k of the contagion algorithm in the network $(C,F)^M$. This holds for any $m_k \in M_k$ and any k.Thus, there exists no rescue merger in M, which completes the proof.

	$(e^M)'$
$N^M = \{1,2,3,4\}$	$(0,0,0,0)$
$N^M = \{\{1,2\},3,4\}$	$(0,0,0,0)$
$N^M = \{\{1,2\},\{3,4\}\}$	$(0,0.0355)$
$N^M = \{1,2,\{3,4\}\}$	$(0,0.0355)$
$N^M = \{\{1,3\},2,4\}$	$(0,0,0,0)$
$N^M = \{\{1,3\},\{2,4\}\}$	$(0,0,0,0)$
$N^M = \{1,3,\{2,4\}\}$	$(0,0,0,0)$
$N^M = \{\{1,4\},2,3\}$	$(0,0,0,0)$
$N^M = \{\{1,4\},\{2,3\}\}$	$(0,0,0,0)$
$N^M = \{1,4,\{2,3\}\}$	$(0,0,0,0)$
$N^M = \{\{1,2,3\},4\}$	$(0,0,0,0)$
$N^M = \{\{1,2,4\},3\}$	$(0,0,0,0)$
$N^M = \{\{1,3,4\},2\}$	$(0,0,0,0)$
$N^M = \{\{2,3,4\},1\}$	$(0.02,0)$
$N^M = \{\{1,2,3,4\}\}$	(0.02)

Table 1. Potential rescue configurations and total payoffs in Example 1[34]

Table 1 depicts the outcomes for potential configurations in Example 1, where $(e^M)'$ is the total payoffs of banks in N^M in each potential partition of set N.

In Example 1, one can see that there exists a strong Nash equilibrium in which banks 3 and 4 form a coalition. In addition, there exists no strong Nash equilibrium in which banks 1 or 2 are involved in a rescue merger. This example illustrates a case of partially prevented contagion where banks in the network have the capability of preventing all failures, which would occur if the grand coalition has been formed, and also have the capability of preventing all potential contagious failures, which would occur if a coalition by banks 2,3, and 4 has been formed.

Moreover, one can see that the results in this example would hold for various sharing rules. As an example, consider the outcome $N^M = \{\{1,2,3,4\}\}$. In this case, even if the sharing rule is such that banks 1 and 2 receives zero payoff, banks 3 and 4 would still deviate and form the coalition of $\{3,4\}$. Similarly, consider the outcome $N^M = \{\{2,3,4\},1\}$. Similarly, even if bank 2 receives zero payoff , banks 3 and 4 would still deviate and form the coalition of $\{3,4\}$.

Proof of Proposition 6:

For given claims in a star network, the matrix $(I-C)$ can be represented as an "arrowhead matrix" such that:

[34]Non-zero entries in Table 1 are rounded up if necessary.

$$I - C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -\alpha_p \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_p \\ 0 & 0 & 1 & \cdots & 0 & -\alpha_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_p \\ -\alpha_c & -\alpha_c & -\alpha_c & \cdots & -\alpha_c & 1 \end{bmatrix}$$

where core node is represented as the n^{th} element of the matrix above.

Following Theorem 2.1 in Najafi et al. (2014), the modified Sherman-Morrison inverse of $(I - C)$ is as follows:

$(I - C)^{-1} = (I - S_1)\left(I - \frac{1}{1+\omega}(S_2(I - S_1))\right)$

where $1 + \omega \neq 0$, $\omega = -(n-1)\alpha_p\alpha_c$, and S_1 and S_2 are strictly lower and strictly upper triangular parts of $(I - C)$ such that $(I - C) = I + S_1 + S_2$.

Then,

$$S_2(I - S_1) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_p \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_p \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_p \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \alpha_c & \alpha_c & \alpha_c & \cdots & \alpha_c & 1 \end{bmatrix}$$

$$I - \frac{S_2(I-S_1)}{1+\omega} = \begin{bmatrix} \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{-\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \ddots & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(I-C)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \alpha_c & \alpha_c & \alpha_c & \cdots & \alpha_c & 1 \end{bmatrix} \begin{bmatrix} \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{-\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \ddots & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$(I-C)^{-1} = \begin{bmatrix} \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{1-(n-2)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \ddots & \frac{\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{1}{1-(n-1)\alpha_p\alpha_c} \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 1-\alpha_c & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & 1-\alpha_c & 0 \\ 0 & \cdots & 0 & 1-(n-1)\alpha_p \end{bmatrix}$$

The dependency matrix A is equal to:

$$A = \hat{C}(I-C)^{-1} = \begin{bmatrix} \frac{(1-\alpha_c)(1-(n-2)\alpha_p\alpha_c)}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)(1-(n-2)\alpha_p\alpha_c)}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \cdots & \ddots & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \cdots & \frac{1-(n-1)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \end{bmatrix}$$

Then, for $p_p = 1$ for all $p \in N \setminus c$, and for given p_c, before any shock hits, we have:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \frac{(1-\alpha_c)(1-(n-2)\alpha_p\alpha_c)}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)(1-(n-2)\alpha_p\alpha_c)}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)(1-(n-2)\alpha_p\alpha_c)}{1-(n-1)\alpha_p\alpha_c} & \frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \\ \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} & \frac{1-(n-1)\alpha_p}{1-(n-1)\alpha_p\alpha_c} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ p_c \end{bmatrix}$$

where bank 4 is the core bank and other banks are the periphery banks. Then, by some lines of algebra,

$v_p = \frac{(1-\alpha_c)(p_p+\alpha_p p_c)}{1-(n-1)\alpha_p\alpha_c}$

$v_c = \frac{(1-(n-1)\alpha_p)(\alpha_c p_p(n-1)+p_c)}{1-(n-1)\alpha_p\alpha_c}$

For $n = 4$ and $p_p = 1$:

$v_p = \left[\frac{(1-\alpha_c)(1+\alpha_p p_c)}{1-3\alpha_p\alpha_c}\right]$ for all $p \in N \setminus c$, and $v_c = \frac{(3\alpha_c+p_c)(1-3\alpha_p)}{1-3\alpha_p\alpha_c}$.

Moreover, the assumption $\sum_{i\in N}(p_i - l_i) \leq p_p$ guarantees that for sufficiently large shocks banks are not capable of preventing all failures. The conditions for the optimal network are as follows.

i) For the core bank, the optimality requires:

$\sum_{i\in N\setminus c} A_{ic} = 1 - A_{cc} = \sum_{i\in N\setminus c}(v_i - l_i)$, which can be rewritten as:

$(n-1)\frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c} = (n-1)\left[\frac{(1-\alpha_c)(p_p+\alpha_p p_c)}{1-(n-1)\alpha_p\alpha_c} - l_p\right]$

By further simplification:

$$l_p = \left[\frac{(1-\alpha_c)(p_p+\alpha_p p_c-\alpha_p)}{1-(n-1)\alpha_p\alpha_c}\right]$$

$$\frac{\alpha_p}{\alpha_c} = \left[\frac{(l_p+p_p(\alpha_c-1))}{\alpha_c(p_c-1-\alpha_c(p_c-1-(n-1)l_p))}\right]$$

For $n = 4$,

$$\frac{\alpha_p}{\alpha_c} = \frac{(l_p+\alpha_c-1)}{\alpha_c(p_c-1-\alpha_c(p_c-1-3l_p))}$$

This gives us the first condition.

ii) For each periphery bank $p \in N \setminus c$, the optimality requires:

$1 - A_{pp} = (v_c - l_c) + \sum_{i \in N \setminus (p \cup c)} (v_i - l_i)$

$1 - A_{pp} = \left[\frac{(1-(n-1)\alpha_p)(\alpha_c p_p (n-1)+p_c)}{1-(n-1)\alpha_p\alpha_c} - l_c\right] + (n-2)\left[\frac{(1-\alpha_c)(p_p+\alpha_p p_c)}{1-(n-1)\alpha_p\alpha_c} - l_p\right]$

Then, I plug $l_p = \left[\frac{(1-\alpha_c)(p_p+\alpha_p p_c-\alpha_p)}{1-(n-1)\alpha_p\alpha_c}\right]$ into above and get:

$1 - A_{pp} = \left[\frac{(1-(n-1)\alpha_p)(\alpha_c p_p (n-1)+p_c)}{1-(n-1)\alpha_p\alpha_c} - l_c\right] + (n-2)\left[\frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c}\right]$

$(n-2)\frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c} + \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} = \left[\frac{(1-(n-1)\alpha_p)(\alpha_c p_p (n-1)+p_c)}{1-(n-1)\alpha_p\alpha_c} - l_c\right] + (n-2)\left[\frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c}\right]$

Then, by rewriting the equation above:

$l_c = -(n-2)(\frac{(1-\alpha_c)\alpha_p\alpha_c}{1-(n-1)\alpha_p\alpha_c}) - \frac{\alpha_c(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c} + \left[\frac{(1-(n-1)\alpha_p)(\alpha_c p_p (n-1)+p_c)}{1-(n-1)\alpha_p\alpha_c}\right] + (n-2)\left[\frac{(1-\alpha_c)\alpha_p}{1-(n-1)\alpha_p\alpha_c}\right]$

$$l_c = \frac{(n-2)(1-\alpha_c)^2\alpha_p + (\alpha_c(p_p(n-1)-1)+p_c)(1-(n-1)\alpha_p)}{1-(n-1)\alpha_p\alpha_c}$$

For $p_p = 1$ and $n = 4$, we get:

$l_c = \frac{2\alpha_p(1-\alpha_c)^2+(2\alpha_c+p_c)(1-3\alpha_p)}{1-3\alpha_p\alpha_c}$

$$\frac{\alpha_p}{\alpha_c} = \frac{(2\alpha_c+(p_c-l_c))}{\alpha_c[(10-3l_c)\alpha_c-2\alpha_c^2+3p_c-2]}$$

This gives us the second condition.

By combining the two equations for $\frac{\alpha_p}{\alpha_c}$, we get:

$$\alpha_p = \frac{2\alpha_c+(p_c-l_c)}{\alpha_c[(10-3l_c)\alpha_c-2\alpha_c^2+3p_c-2]} = \frac{(l_p+\alpha_c-1)}{\alpha_c(p_c-1-\alpha_c(p_c-1-3l_p))}$$

which gives us a solution for α_c as a function of $p_c, l_c, p_p,$ and l_p. By plugging $\alpha_c^*(p_c, l_c, p_p, l_p)$ into conditions above, we get $\alpha_p^*(p_c, l_c, p_p, l_p)$.

Lastly, for each bank to be initially solvent before any shock hits, it should be that $v_c \geq l_c$ and $v_p \geq l_p$. First, as one can see from the equations above, $v_p \geq l_p$ always holds. Second, $v_c \geq l_c$ holds iff $\frac{\alpha_c(1-3\alpha_p)}{2\alpha_p(1-\alpha_c)^2} \geq 1$ holds, which is the third condition.

These three conditions together characterize the solution for star network.

Then there exists (p_c, l_c, p_p, l_p) such that a star network with claims $\alpha_c^*(p_c, l_c, p_p, l_p)$ and $\alpha_p^*(p_c, l_c, p_p, l_p)$ satisfies these initial conditions for the optimality.

Next, I show that in such star network with $\alpha_c^*(p_c, l_c, p_p, l_p)$ and $\alpha_p^*(p_c, l_c, p_p, l_p)$, whenever there is a distressed bank, the grand coalition is formed at equilibrium (with or without government intervention, depending on the shock level).

Under the assumption that $\sum_{i \in N}(p_i - l_i) \leq \frac{1}{2}$, it is always true that whenever a periphery bank is hit by a shock and becomes distress, then the core bank and the other periphery banks also become distressed. Similarly, whenever the core bank is hit by the shock and becomes distressed, then under the assumption that $p_c > 1$, all other banks are distressed banks as well because $A_{ic} = v_i - l_i$ for all $i \in N \setminus c$ implying that $A_{ic}p_c > v_i - l_i$ for all $i \in N \setminus c$.

Next, in the given star network, $\sum_{i\in N}(p_i - l_i) \leq \frac{1}{2}$ implies that any single default would cause a total loss at least as much as the total available capital in the system. Moreover, whenever the shocked bank is in distress, any coalitional deviation that involves the shocked bank would default due to the shock. Any coalitional deviation that does not involve the shocked bank would default. Therefore, whenever any coalition that does not involve some banks and if those banks default, then any other coalition among the remaining banks would also default. Therefore, similar to the previous proofs, it follows form here that the grand coalition is equilibrium (with or without government assistance, depending on the shock level) whenever there exists any distressed bank following a given shock.

Example 7.

I set $p_p = 1$, $p_c = 5$, $l_p = \frac{42}{44} < 1$, $l_c = \frac{1033}{220} < 5$.

Then, $\alpha_p = 0.1$ and $\alpha_c = 0.4$ is an optimal network, where $v_c = \frac{1085}{220}$ for the core bank, and $v_p = \frac{45}{44}$ for each periphery bank, and the dependency matrix is

$$A = \begin{bmatrix} \frac{69}{110} & \frac{3}{110} & \frac{3}{110} & \frac{3}{44} \\ \frac{3}{110} & \frac{69}{110} & \frac{3}{110} & \frac{3}{44} \\ \frac{3}{110} & \frac{3}{110} & \frac{69}{110} & \frac{3}{44} \\ \frac{14}{44} & \frac{14}{44} & \frac{14}{44} & \frac{35}{44} \end{bmatrix}$$

Proof of Theorem 3:

The dependency matrix and vector $\vec{v}$ is given by:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

Step 1) First, the Assumption of $\sum_{i=1}^{n}(p_i - l_i) < \frac{1}{2} < min\{p_1, ..., p_n\}$ implies that the shock to any bank can be large enough that banks are not capable of preventing all failures. In such a case, the government assistance is required to prevent all failures. Whenever the government decides to assists, the level of assistance does not depend on the level of connectivity but only depends on $\sum_{i=1}^{n}(p_i - l_i)$, which is exogenously given. Whenever, the government decides not to rescue, then the rest of banks can prevent all potential contagious failures if $1 - A_{kk} \leq \sum_{j\in N\setminus k}(v_j - l_j)$. Therefore, similar to the proof of Theorem 2 and Proposition 5, for each bank k, the following property is the first desired property.

$$(1 - A_{kk})\, p_k = \sum_{j\in N\setminus k}(v_j - l_j)$$

where $(1 - A_{kk})$ measures the total dependency of other banks to bank k. In this case, failure of a bank causes a loss to the rest of the system equal to the total net worth of remaining banks.

By using $\hat{v} = A\hat{p}$, the expression can be rewritten as:

$$\sum_{i \in N \setminus k} \sum_{j \in N \setminus k} A_{ij} p_j = \sum_{j \in N \setminus k} l_j \text{ for all } k \in N \ (k \text{ is a singleton})$$

This equation implies that the dependency of other banks to bank k, $(1 - A_{kk})$, is at intermediate levels since it is a function of $\hat{p}$ and $\hat{l}$, and it is always greater than zero.

Step 2) For $(1 - A_{kk})\, p_k = \sum_{j \in N \setminus k} (v_j - l_j)$ for all $k \in N$ (k is a singleton), it holds that whenever the shocked bank (a randomly chosen bank) is distressed either

i) there exists at least one extra bank that is distressed (there exists a bank $j \in N \setminus r$ s.t. $(1 - A_{jr})\, p_r > v_j - l_j$), or

ii) $(1 - A_{jr})\, p_r = (v_j - l_j)$ holds for each $j \in N \setminus r$ (net worth of each bank in the set of $\{N \setminus r\}$ drops to zero, as in the complete network case).

If the second condition holds (such as in complete network), then we are done. If the first condition holds, move to step 3.

Step 3) Next, I show that whenever the condition below holds and the banks do not have capability to rescue the shocked bank, then there exists no group of banks that can deviate from grand coalition and have a shareholders' value greater than zero without government support.

$$\sum_{i \in K} \sum_{j \in K} A_{ij} p_j \leq \sum_{i \in K} l_i \text{ for all } K \subset N$$

Any $K \subset N$ defaults. For any $K \subset N \setminus r$, the condition above implies that either

i) $\sum_{j \in N \setminus k} A_{kr} p_r > \sum_{j \in K} (v_j - l_j)$, otherwise

ii) there exists a bank in $j \in N \setminus \{K \cup r\}$ such that $A_{jr} p_r > v_j - l_j$. Then, for any consequent steps the similar scenario holds, and eventually the loss emanating from failing banks outside of group K exceeds $\sum_{j \in K} (v_j - l_j)$. So, any coalition that does not involve bank $r$also defaults, so deviation is not profitable.

Next, consider the small shocks where banks can coalitionally prevent all failures. In such a case, the sharing rule guarantees that there exists no profitable deviation by a coalition smaller than the grand coalition that involves bank r. The reason is that for a deviation to be profitable it should be that any remaining bank must be solvent, otherwise as shown above the total loss exceed $\sum_{j \in K} (v_j - l_j)$ and the coalition defaults. Consider a partition $\{K, N \setminus K\}$ such that coalition K involves bank r and can absorb the shock, and the remaining banks are solvent. Then, given the sharing rule the grand coalition gives at least as much as payoff as each bank in coalition K receives in partition $\{K, N \setminus K\}$. Thus, the grand coalition becomes an equilibrium outcome.

Any financial network (C, F) that does not satisfy these two properties is not a social-welfare maximizing network. First, whenever part i) does not hold and the integration level for any bank is greater than the given intermediate level of integration, then this implies that for sufficiently large shocks, the government has to step in to maintain the stability in the rest of the system that drops the social welfare. On the other hand, if there exists a bank that has a lower integration level than the given intermediate level of integration, then the contributions of other banks in rescue of that bank would be lower, which again drops the social welfare.

Second, whenever part ii) does not hold, then as discussed above there always exists a coalition that can deviate from the grand colation and end up with a value greater than zero, which implies that there exist cases where the welfare is less than the first-best case.

Lastly, the characterization result here relies on the dependency matrix A but the existence of such networks relies on the claims matrix C where $A = \hat{C}(I - C)^{-1}$. Lastly, Lemma 1 together with the conditions given in Theorem 3 imply that there always exists a C matrix where $A = \hat{C}(I - C)^{-1}$ satisfies the conditions in Theorem 3. This finalizes the proof.

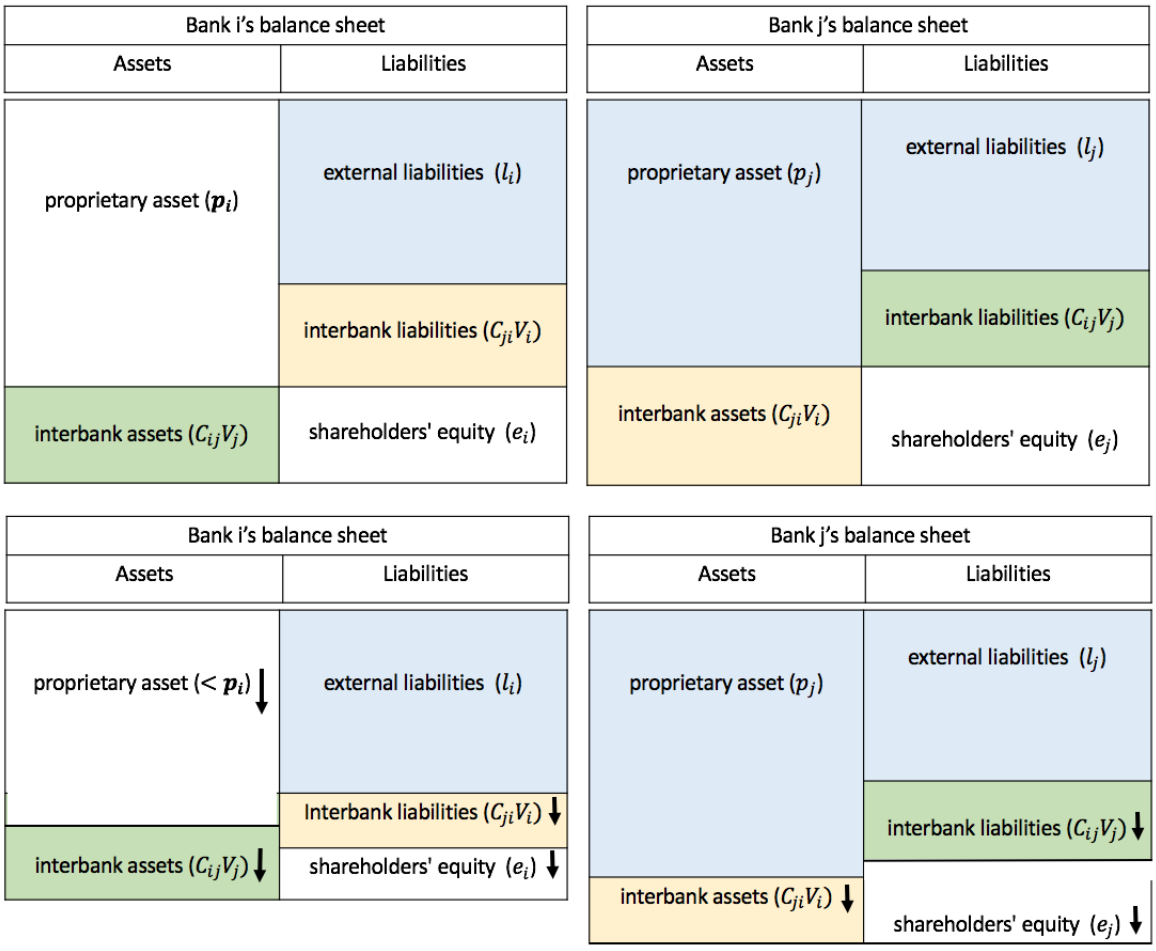

Figure 13: Interdependencies in balance sheets

The Linearities in Interbank Contracts.—Figure 13 illustrates how a decline in the proprietary asset return of bank i affects the interbank liabilities and the net worth of bank i and j.

Figure 13 shows the linearities in interbank assets and liabilities. Such linearities might be thought of as the voluntary "write-downs" of interbank liabilities in distressed times. For instance, if any bank faces a sharp reduction in mortgage loan repayments by households, then its assets decrease and the claim structure implies that its interbank liabilities also drop at some rate. Given that the external liabilities always remain fixed, the drop in its total liabilities is always smaller than the drop in its total assets. This feature makes the model with bankruptcies robust to the feature of linearities. Equation (3) shows that when the total assets of bank i (V_i) decreases by one unit, its interbank liabilities decrease in amount of $\sum_{j \in N} C_{ji} < 1$ units.

7.1 Discussion on Cooperative Game and Non-Cooperative Game Settings

The coalition formation framework built on a contagion model has two specific features. First, bank rescues are similar to public goods, where all banks weakly gain some benefits from any rescue that prevent a costly bankruptcy, whereas rescuer banks pay some costs. The (potential) costs for rescuers might exceed the (potential) benefits in some cases. Secondly, rescue mergers are complements to each other and, hence, the superset of a given set of rescue mergers is also a rescue merger, for everything else constant. These two specifications together imply that both a cooperative game setting and a non-cooperative game setting would work similarly and provide the same insights and results about welfare analysis. Similarly, among non-cooperative game settings, solution concepts of strong Nash equilibrium and coalition-proof Nash equilibrium provide the same results. In the paper, I define a non-cooperative game for rescue formation and use strong Nash equilibrium as a solution concept. Alternatively, a cooperative (or coalitional) game setting could be used under proper *core* definitions that captures the externalities[35] and the possibility of formation of coalitions smaller than the grand coalition.

[35]Partition Function Games (PFGs) are the cooperative games that capture the externalities in a given economy, different than the Characteristics Function Games. In a PFG, payoffs of agents in a coalition depends on the partition of the economy, so depends on what other agents do.