Optimal prizes in tournaments under nonseparable preferences*

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Abstract

We study rank-order tournaments with risk-averse agents whose utility over money and effort (or leisure) may be nonseparable. We characterize optimal prize schedules when the principal allocates a fixed budget and show how they are determined by the interplay between the properties of noise and the utility function. In particular, the distribution of noise alone determines whether the optimal prize schedule has flat regions where some number of prizes are equal, while the total number of positive prizes depends on both the noise distribution and utility. For unimodal noise distributions, the optimal number of positive prizes is restricted regardless of utility under mild assumptions. Also, while the common wisdom suggests—and it holds in the separable case—that risk aversion pushes optimal prize allocations in the direction of prize sharing, this is no longer true, in general, when the marginal utility of money depends on effort.

Keywords: tournament, optimal allocation of prizes, risk aversion, nonseparable utility, majorization.

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1 Introduction

Many incentive systems are structured as *rank-order tournaments*. In organizations, managers use promotions, salary raises and bonuses to reward the best employees.[1] At schools and universities, instructors often assign grades "on a curve," effectively creating a rank-based reward scheme. Firms using forced rankings are still pretty common, despite a significant controversy surrounding their effectiveness. In many of these settings, employee effort is not directly observable; moreover, productivity measures that serve as the basis for rankings have a significant luck component. The prototypical example is a sales contest where a salesperson's output depends on the individual's effort as well as a realization of random factors beyond one's control, such as the arrival of agreeable clients.

One of the central questions in the design of incentive systems, including tournaments, is how to optimally structure rewards. With rank-based rewards, the most obvious approach to consider is the winner-take-all (WTA) scheme where the winner—the best-performing employee—receives the entire prize. However, there may be reasons for various degrees of *prize sharing*—a more equitable allocation of prizes—to work better. One commonly cited such reason is risk aversion: A risk-averse employee would be willing to work harder if she knew that even in the unlucky event of a bad shock she can still receive some reward for being ranked second, third, etc. The gain in the marginal benefit of effort from such insurance can more than offset the loss due a reduction in the top prize, leading to the optimality of prize sharing. This intuition also suggests that as agents become more risk-averse, more prize sharing would be needed to maximize incentives.

In this paper, we study the optimal allocation of prizes in tournaments for risk-averse agents. The existing literature (discussed in more detail below) has focused mostly on the separable case, with agents' utility for money-effort allocations (v, e) of the form $U(v, e) = u(v) - c(e)$, where $c(\cdot)$ is an effort cost function, and $u(\cdot)$ is a Bernoulli utility function. The results of these models confirm the intuition about the effect of risk aversion on optimal prize sharing. Our major innovation is in considering a substantially more general class of nonseparable utility functions $U(v, e)$.

Nonseparable preferences over consumption and leisure are standard in micro-founded models of labor supply (e.g., Killingsworth, 1983; Pencavel, 1986), real business cycle, growth, and policy analysis (e.g., Blanchard and Fischer, 1989; Romer, 2019).[2] The

[1] Rewarding top employees is seen as key to the firm's success by HRM practitioners. See e.g., https://www.shrm.org/hr-today/news/hr-magazine/pages/0914-rewards-performance-based-pay.aspx.

[2] For example, nonseparability helps explain a number of empirical puzzles in international trade (Lewis, 1996; Jermann, 2002; Matsumoto, 2009) and fiscal policy (Linnemann, 2006; Bilbiie, 2011).

most popular preferences used in macroeconomics are, arguably, those introduced by Greenwood, Hercowitz and Huffman (1988), $U(v - c(e))$, and by King, Plosser and Rebelo (1988), $\frac{v^{1-\alpha}}{1-\alpha}c(e)$. It is, therefore, natural to extend the existing analysis of incentives, including rank-order tournaments, to nonseparable preferences over money and effort (which can be viewed as a decreasing function of leisure).

We use a simple model of a tournament à la Lazear and Rosen (1981) and assume that agents are identical in their ability and risk preferences. These simplifications allow us to focus on the properties of noise and (homogeneous) features of utility to explore the effects of nonseparability. It can also be argued that tournament incentive schemes are more effective, and hence are more likely to be used in practice, in settings where agents are not too different in their characteristics.

We first characterize the optimal—effort-maximizing—allocation of prizes under relatively mild assumptions. The optimal prize schedule is determined by the properties of noise, as well as the shape of utility. Generically, optimal prizes have a step-wise structure. There is a sequence of *critical ranks*, determined only by the distribution of noise, such that prizes decline at the critical ranks and remain constant in between, as long as they are positive. To understand why a step-wise structure may emerge, consider a symmetric equilibrium in the tournament. In a "well-behaved" model, the expected marginal utility of effort is declining, and it is equal to zero in equilibrium. The maximum achievable effort is, therefore, determined as the root of the *marginal utility frontier*, which is obtained by maximizing expected marginal utility over all feasible prize allocations at each symmetric level of effort. In the symmetric equilibrium, each of the n players is equally likely to end up with any rank 1 through n; therefore, the expected marginal utility is determined by a sequence of marginal probabilities of reaching ranks $1, \ldots, n$. When this sequence is monotonically decreasing—as would be the case, for example, for log-concave noise distributions—so is the sequence of optimal prizes, because it is optimal to provide stronger incentives for higher ranks. But if the sequence of marginal probabilities is nonmonotone—as may be the case when the distribution of noise is multi-modal or has a heavy tail—it would have been optimal to assign prizes to ranks nonmonotonically if such prize schedules were feasible. A natural monotonicity requirement for prizes then leads to a step-wise prize structure via an "ironing" procedure.

The optimal number of positive prizes—in particular, whether or not WTA is optimal—is determined jointly by the properties of noise and utility. To see the effect of nonseparability, consider first the risk-neutral case analyzed by Drugov and Ryvkin (2020). There, a generic optimal prize allocation is a two-prize schedule with several equal prizes at the top and zero prizes for lower ranks. The number of top prizes is determined by the noise distribution. For risk-averse agents, the smoothing of marginal returns to different prizes calls for a more gradually declining prize structure, and more prize sharing. As a result, the step-wise structure mentioned above emerges, where the first step is located at the same rank as the (only) step in the risk-neutral case.

When risk-averse agents' preferences are separable, the optimal prize schedule maximizes the expected utility of prizes weighted by the marginal probabilities of reaching the corresponding ranks. Since the last marginal probability is always negative—a higher effort always decreases the probability of being ranked last—rank n is never assigned a positive prize. Depending on the noise distribution, several ranks before the last one may also have negative marginal probabilities of being reached, and hence never have positive prizes assigned to them as well. For some noise distributions—such as uniform—only the first marginal probability is positive, which leads to WTA being optimal for *any* separable utility function.

However, when the utility function is nonseparable, a new effect comes into play: Prizes now affect the marginal cost of effort. Hence, the optimal prize schedule strikes a balance between maximizing the marginal utility of prizes and minimizing the marginal cost of effort. The direction of the latter effect depends on the utility function. When money and effort are complements—that is, the cross derivative U_{ve} is positive—even more prize sharing becomes optimal, to the extent that in some cases it is optimal to give positive prizes to *all* ranks, including the last place. When money and effort are substitutes, $U_{ve} \leq 0$, the "regular" behavior is restored. The last place is never assigned a positive prize; moreover, as long as the distribution of noise is unimodal, there is a restriction on the number of positive prizes (and hence, on the degree of prize sharing) regardless of risk aversion.

Having characterized optimal prizes, we then study how an increase in risk aversion affects the optimal prize schedule. To this end, we introduce a generalized definition of risk aversion for nonseparable utility functions. This definition includes the standard Arrow-Pratt definition—in terms of relative curvature U_{vv}/U_v—and adds another condition in terms of relative complementarity of money and effort, U_{ve}/U_v. In general, there are limits to what can be said about the effects of risk aversion on optimal prize sharing in this setting. This is because the implemented effort depends on prizes, and prizes are directly affected by effort. When utility is transformed, the optimal implemented effort changes, and hence the resulting changes in optimal prizes are essentially unrestricted. However, similar to the concept of compensated demand in classic demand theory, it is possible to introduce *compensated* utility changes. These are changes that preserve the implemented optimal effort, similar to how compensated price changes in demand theory preserve the level of the consumer's utility.

We provide sufficient conditions for when an increase in (generalized) risk aversion leads to more prize sharing. These conditions are easiest to interpret for utility functions of the form $U(v, e) = u(v - c(e))$ transformed by an increasing transformation $\phi(\cdot)$. For compensated utility changes, the result holds when both $u(\cdot)$ and $\phi(\cdot)$ have non-increasing absolute risk aversion (NIARA). For uncompensated utility changes, additional restrictions are needed, one of them being that $\phi(\cdot)$ has non-increasing absolute prudence (NIAP). The spirit of these conditions is a restriction on the change of higher-order cross derivatives of U to make sure that they do not counterbalance the main effect of a higher risk aversion.

Results for separable preferences are obtained as a special case. Those results do not require compensation, similar to how compensated and uncompensated demand are equal when preferences are quasi-linear. We generalize all the existing results for the separable case and show that the effect of risk-aversion on prize sharing is indeed universal for this class of utility functions for any noise distribution. For CARA and CRRA utility functions, the optimal allocation of prizes can be obtained in a closed form.

Relation to the existing literature. This paper contributes to the literature on the optimal allocation of prizes in contests. Two types of models of (static) multi-prize contests have been used historically in the literature, differing mostly in their assumptions about the winner determination process. Noisy, or imperfectly discriminating, contest models are the rank-order tournament model of Lazear and Rosen (1981) and the nested contest model of Clark and Riis (1996), which is a multi-prize adaptation of the classic rent-seeking model of Tullock (1980).[3] The main feature of these models is the presence of idiosyncratic *noise*, or luck, in the transformation of agents' effort into output, and hence a higher effort does not guarantee a higher rank (and a higher prize). In contrast, perfectly discriminating contest models (e.g., Baye, Kovenock and De Vries, 1996; Moldovanu and Sela, 2001; Siegel, 2009; Fang, Noe and Strack, 2020) are all-pay auctions with a one-to-one assortative mapping of effort rankings to prizes. The mechanisms underlying incentives in the two types of models are different, and hence they produce diverging results regarding optimal prizes. In auctions, bids and prizes are typically monetary and measured in the

[3] Under risk-neutrality, the latter model can be written as a special case of the former (Fu and Lu, 2012).

same dimension. Hence, it is the nonseparability of money and type which is typically studied as in, e.g., Maskin and Riley (1984).[4]

Our paper is the first to study allocation of prizes under nonseparable preferences in rank-order tournaments. Previous studies devoted to the effect of risk aversion (Green and Stokey, 1983; Nalebuff and Stiglitz, 1983; Krishna and Morgan, 1998; Kalra and Shi, 2001; Akerlof and Holden, 2012) all consider separable preferences. Among the closest papers, Krishna and Morgan (1998) assume that the principal allocates a fixed budget and show that WTA is always optimal in tournaments of size $n \leq 3$ for risk-averse agents if the distribution of noise is unimodal and symmetric (the result is referred to as the "winner-take-all principle" for small tournaments). We show that unimodality is not required and provide conditions for the optimality of WTA in tournaments with $n \leq 3$ for nonseparable utility. Kalra and Shi (2001) find optimal prize schedules under logistic and uniform noise distributions. Both are log-concave and, as we show, for this class of distributions the monotonicity constraint on prizes is not binding. We provide a general characterization both in this case and in the more interesting case when the noise distribution is not log-concave, the constraint binds, and an "ironing" procedure is needed. Akerlof and Holden (2012) consider a tournament with risk-averse agents where the prize budget is not fixed but the agents' participation constraint is binding. They show that various patterns of prize sharing are optimal depending on the agents' risk-aversion and prudence, but the results are too convoluted to discern the effects of the properties of noise or general conditions for the optimality of WTA.

In the Tullock contest setting, the only form of nonseparability studied in the literature is $U(v, e) = u(v - e)$.[5] Fu, Wang and Wu (2021) and Liu and Treich (2021) study the optimal allocation of prizes in this setting and show that prudence—that is, the sign and magnitude of $u'''(\cdot)$—plays an important role.[6] In this paper, we consider more general nonseparable preferences $U(v, e)$ and show that the sign of the second and third cross derivatives, U_{ve} and U_{vve}, is often important. The latter reduces to prudence when

[4]A separate strand of literature explores "large" contests where each player takes the distribution of actions of others as given, and the equilibrium is defined as a self-consistent distribution maximizing individual payoffs (Glazer and Hassin, 1988; Olszewski and Siegel, 2016).

[5]The paper by Dickson, MacKenzie and Sekeris (2018) is an exception in considering a general utility function $U(v, e)$. Taking the share contest interpretation of the Tullock setting, they show that efforts may decrease when the total rent is higher.

[6]Other papers, such as Hillman and Katz (1984), Konrad and Schlesinger (1997), Cornes and Hartley (2003), Treich (2010), Cornes and Hartley (2012) and Schroyen and Treich (2016), focus on equilibrium existence conditions (which we also provide) and the effect of risk aversion on effort. Wang, Wu and Zhu (2020) compare centralized and decentralized WTA contests in terms of efforts.

$U(v, e) = u(v - e)$. The importance of non-increasing absolute prudence (NIAP) is new, to the best of our knowledge.

Finally, our paper generalizes the analysis of the optimal prize schedules in rank-order tournaments with risk neutral players. Drugov and Ryvkin (2020) show that WTA is optimal when the hazard rate of noise is increasing, whereas giving equal prizes to all but the agent ranked last is optimal when it is decreasing; they also provide some results for nonmonotone hazard rates.[7] These results generalize the findings of Clark and Riis (1996) and Schweinzer and Segev (2012) who showed the optimality of WTA for nested Tullock contests whose equilibrium is isomorphic to that of a tournament with additive noise with the Gumbel distribution (which has an increasing hazard rate).

The rest of the paper is organized as follows. Section 2 sets up the model and provides some preliminary steps. Section 3 derives the optimal prize schedule. The effect of risk aversion is considered in Section 4. Section 5 specializes the general results to the case of the separable utility. The connections and implications for Tullock contests are discussed in Section 6. Section 7 concludes. The conditions for the equilibrium existence are provided in Appendix A. All proofs are contained in Appendix B.

2 Model setup

We consider a tournament of $n \geq 2$ identical agents indexed by $i \in \mathcal{I} = \{1, \ldots, n\}$. The agents simultaneously and independently choose effort levels $e_i \in \mathbb{R}_+$. The output of agent i is her effort perturbed by additive noise, $Y_i = e_i + X_i$.[8] Shocks X_i are zero-mean, i.i.d. copies of random variable X that has an absolutely continuous cumulative distribution function (cdf) $F(\cdot)$ and probability density function (pdf) $f(\cdot)$ defined on an interval support $\mathcal{X} = [\underline{x}, \overline{x}]$ (where $\underline{x}$ and $\overline{x}$ can be finite or infinite). The pdf $f(\cdot)$ is continuous, piece-wise differentiable, and square-integrable.

A risk-neutral principal observes the ranking of outputs and allocates rank-dependent prizes $\mathbf{v} = (v_1, \ldots, v_n)$ to the n agents. Specifically, an agent whose output is ranked r (where $r = 1$ corresponds to the highest output, $r = 2$ to the second highest, etc.) receives

[7] Krishna and Morgan (1998) and Ales, Cho and Körpeoğlu (2017) provide some partial results in this setting. Balafoutas et al. (2017) show that prize sharing can be optimal if agents are heterogeneous even when they are risk-neutral.

[8] Via a change of variables, this model can also accommodate tournaments with multiplicative noise, with $Y_i = e_i X_i$ (see Jia, 2008; Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2020).

a prize v_r.[9] Prizes are nonnegative, decreasing[10] in rank, $v_1 \geq v_2 \geq \ldots \geq v_n \geq 0$, and satisfy the budget constraint $\sum_{r=1}^n v_r = b > 0$. We will refer to prize vectors $\mathbf{v}$ satisfying these conditions as *feasible*, and use $\mathcal{V}_b$ to denote the set of all such vectors.

Agents have the same utility function $U : [0, b] \times \mathbb{R}_+ \to \mathbb{R}$ over money and effort pairs (v, e). We assume that U is a C^3 function, except, possibly, at $(0, e)$, where it is continuous in v, and at $(v, 0)$, where it is C^1 in e. Furthermore, $U_v > 0$, $U_{vv} \leq 0$, $U_e < 0$ for $e > 0$, and $U_{ee} < 0$. Finally, $U_e(v, 0) = 0$ and, without loss, $U(0, 0) = 0$. We will refer to utility functions satisfying these assumptions as *regular*. Two important special cases are (i) $U(v, e) = \eta(v - c(e))$, where $\eta : \mathbb{R} \to \mathbb{R}$ is concave and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly convex cost function satisfying $c(0) = c'(0) = 0$; and (ii) the fully separable case (considered in detail in Section 5), $U(v, e) = u(v) - c(e)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is a concave Bernoulli utility function of money and c is as in (i).

Fix a prize vector $\mathbf{v} \in \mathcal{V}_b$. We look for a symmetric pure-strategy Nash equilibrium where $e_i = \hat{e}$ for all $i \in \mathcal{I}$. Assuming that all but one indicative agent choose effort $\hat{e}$, the expected utility of the indicative agent from some deviation effort e is

$$W(e, \hat{e}) = \sum_{r=1}^{n} \pi^{(r)}(e, \hat{e}) U(v_r, e), \tag{1}$$

where $\pi^{(r)}(e, \hat{e})$ is the probability that the indicative agent's output is ranked r. This probability is given by[11]

$$\pi^{(r)}(e, \hat{e}) = \binom{n-1}{r-1} \int F(e - \hat{e} + x)^{n-r} [1 - F(e - \hat{e} + x)]^{r-1} dF(x). \tag{2}$$

Indeed, in order to be ranked r, the indicative agent's output must be higher than the output of exactly $n - r$ other agents, and there are $\binom{n-1}{r-1}$ ways to choose those agents.

The symmetric first-order condition, $W_e(\hat{e}, \hat{e}) = 0$, produces the equation

$$A(\mathbf{v}, \hat{e}) \equiv \sum_{r=1}^{n} \left[\beta_r U(v_r, \hat{e}) + \frac{1}{n} U_e(v_r, \hat{e}) \right] = 0, \tag{3}$$

[9]Ties in the ranking occur with zero probability for an atomless $f(\cdot)$; therefore, we do not need to specify a tie-breaking rule.

[10]Throughout this paper, "increasing" means nondecreasing and "decreasing" means nonincreasing. Whenever the distinction is important, we use the terms "strictly increasing" and "strictly decreasing."

[11]Integration is over $\mathcal{X}$ unless noted otherwise.

where $\beta_r \equiv \pi_e^{(r)}(\hat{e}, \hat{e})$, the marginal probabilities of reaching rank r, are given by

$$\beta_r = \binom{n-1}{r-1} \int F(x)^{n-r-1}[1-F(x)]^{r-2}[n-r-(n-1)F(x)]f(x)dF(x). \tag{4}$$

Coefficients β_r are determined entirely by the distribution of noise. The following lemma summarizes some of their properties.

Lemma 1 *(i)* $\sum_{r=1}^{n} \beta_r = 0$, $\beta_1 > 0$, $\beta_n < 0$; *if* $f(\cdot)$ *is symmetric, then* $\beta_r = -\beta_{n-r+1}$ *for all* r.
(ii) If $f(\cdot)$ *is (strictly) log-concave, then* β_r *is (strictly) decreasing in* r.
(iii) If $f(\cdot)$ *is (strictly) log-convex and* $f(\overline{x}) = 0$, *then* β_r *is (strictly) increasing in* r *for* $r \leq n-1$.
(iv) If $f(\cdot)$ *is first log-concave, then log-convex with* $f(\overline{x}) = 0$, *then* β_r *is unimodal in* r.
(v) If $f(\cdot)$ *is unimodal, then* β_r *is single-crossing* $+-$; *that is, there exists an* $\hat{r} \leq n-1$ *such that* $\beta_r > 0$ *for* $r \leq \hat{r}$ *and* $\beta_r \leq 0$ *for* $r > \hat{r}$.

If $\hat{e}$ solving (3) exists, it is a natural candidate for a symmetric equilibrium. In Proposition A1 in Appendix A, we provide sufficient conditions for (3) to have a unique solution, and for that solution to be the equilibrium. We also ensure that the participation constraint, $W(\hat{e}, \hat{e}) \geq U(0,0) = 0$, is never binding. Intuitively, the existence conditions require that there is "enough noise" in the tournament, and the cost of effort is sufficiently convex. Importantly, there are no restrictions on the *shape* of the distribution of noise, such as MLRP (or log-concavity), which are typical for the existence of optimal contracts in the standard moral hazard problem. Thus, unlike in the standard moral hazard model, heavy-tailed shocks are admissible here.

In what follows, we will assume that the assumptions of Proposition A1 are satisfied.

3 The structure of optimal prizes

We study the principal's problem

$$\max_{\mathbf{v},e} e \quad \text{s.t. } A(\mathbf{v}, e) = 0, \quad \mathbf{v} \in \mathcal{V}_b. \tag{5}$$

Under our assumptions for the equilibrium existence (Proposition A1), the equation $A(\mathbf{v}, e) = 0$ has a unique solution $\hat{e}(\mathbf{v})$ for each $\mathbf{v} \in \mathcal{V}_b$. Moreover, function $A(\mathbf{v}, e)$ is continuous in $(\mathbf{v}, e)$ and strictly decreasing in e; therefore, by the implicit function theorem for strictly monotone functions (Jittorntrum, 1978), $\hat{e}(\mathbf{v})$ is continuous. The compactness of $\mathcal{V}_b$ then implies, by the Weierstrass theorem, that problem (5) has a solution $(\mathbf{v}^*, e^*)$, with $e^* = \hat{e}(\mathbf{v}^*)$. By construction, e^* is unique.

Function $A(\mathbf{v}, e)$ represents an agent's marginal utility of effort when all agents exert effort e. In the symmetric equilibrium, this marginal utility is equal to zero and decreasing in e; therefore, the largest effort e^* such that $A(\mathbf{v}, e^*) = 0$ is the unique root of the frontier $\max_{\mathbf{v}\in\mathcal{V}_b} A(\mathbf{v}, e)$—the maximum marginal utility that can be reached at symmetric effort e. Formally, consider the problem

$$\max_{\mathbf{v}} A(\mathbf{v}, e) \quad \text{s.t. } \mathbf{v} \in \mathcal{V}_b, \tag{6}$$

and let $\mathbf{v}^*(e)$ denote its (possibly multi-valued) solution, which is well-defined for each $e \in [0, \bar{e}]$ because $A(\mathbf{v}, e)$ is continuous and $\mathcal{V}_b$ is compact.

Proposition 1 *(i) Suppose $(\mathbf{v}^*, e^*)$ is a solution to problem (5). Then $\mathbf{v}^* \in \mathbf{v}^*(e^*)$.*

(ii) There exists a unique $e^* \in [0, \bar{e}]$ such that $(\mathbf{v}^*, e^*)$ is a solution to problem (5) for all $\mathbf{v}^* \in \mathbf{v}^*(e^*)$.

Proposition 1 shows that we can essentially separate the properties of optimal prize structures from finding the optimal effort. It clearly works in the separable case where the marginal benefit of effort depends only on $\mathbf{v}$ while the marginal cost only on e, which implies the optimal prize schedule is independent of e^* (see Section 5). In the nonseparable case, in general, the optimal prize schedule ultimately depends on e^*; however, to explore its properties we can analyze problem (6), with $e = e^*$, treating e^* as a parameter.

Moreover, part (ii) of Proposition 1 shows that problem (6) completely characterizes optimal prize schedules because, once it is solved for all e, the unique optimal e^* can be found as the root of $A(\mathbf{v}^*(e), e)$.[12]

3.1 Optimal prizes without the monotonicity constraint

We start by analyzing a relaxed version of problem (6), with $e = e^*$, where the monotonicity constraint is dropped:

$$\max_{\mathbf{v}} A(\mathbf{v}, e^*), \quad \text{s.t. } \sum_{r=1}^{n} v_r = b, \quad v_1, \ldots, v_n \geq 0. \tag{7}$$

[12] Proposition 1(ii) also implies the existence of a solution to problem (5) directly from the existence of a solution to problem (6). There is no need to invoke the implicit function theorem discussed in the first paragraph of Section 3.

We will identify conditions under which solutions to problem (7) are, in fact, monotone, and hence they also solve (6). We will then use the features of the solution to (7) to construct a solution in the general case.

Using the definition of function $A(\mathbf{v}, e)$, Eq. (3), the Lagrangian for problem (7) can be written as

$$\mathcal{L}^{(0)}(\mathbf{v}, \lambda; e^*) = \sum_{r=1}^{n} \left[\beta_r U(v_r, e^*) + \frac{1}{n} U_e(v_r, e^*) - \lambda v_r \right] + \lambda b, \tag{8}$$

which gives the Kuhn-Tucker (KT) conditions,

$$\beta_r U_v(v_r, e^*) + \frac{1}{n} U_{ve}(v_r, e^*) \leq \lambda, \quad \text{with equality if } v_r > 0, \quad r = 1, \ldots, n. \tag{9}$$

The constraint in problem (7) is linear; therefore, by the KT necessity theorem, for any solution $\mathbf{v}^*$ to (7) there exists a Lagrange multiplier $\lambda^* > 0$ such that $\mathbf{v}^*$ satisfies (9) and the budget constraint.

The following correspondence plays a key role in the structure of optimal prizes:

$$q(\beta, \lambda; e^*) = \arg\max_{v \geq 0} \left[\beta U(v, e^*) + \frac{1}{n} U_e(v, e^*) - \lambda v \right]. \tag{10}$$

For any $\lambda > 0$, the objective in (10) is continuous, bounded above, and goes to $-\infty$ for $v \to \infty$; therefore, $q(\beta, \lambda; e^*)$ is well-defined. From the KT necessity theorem, any $v_r \in q(\beta_r, \lambda; e^*)$ satisfies (9). Moreover, any solution $\mathbf{v}^*$ to (7) maximizes $\mathcal{L}^{(0)}(\mathbf{v}, \lambda^*; e^*)$. As seen from (8), the Lagrangian is additive separable in prizes, and hence $v_r^* \in q(\beta_r, \lambda^*; e^*)$. The optimal Lagrange multiplier λ^* satisfies the budget constraint $\sum_{r=1}^{n} q(\beta_r, \lambda; e^*) = b$. (MISHA: I removed the notation of $Q(\lambda; e^*)$ since we use it only once here).

Intuitively, coefficient β_r represents the marginal effect of effort on the probability of being ranked r in the symmetric equilibrium. When prizes are unrestricted, it is, therefore, optimal to assign higher prizes to ranks with higher β_r. Therefore, when β_r is decreasing in r, such an assignment automatically satisfies the monotonicity constraint. This gives our first major result.

Proposition 2 *If $f(\cdot)$ is log-concave, there exists a solution $\mathbf{v}^*$ to the relaxed problem (7) that also solves problem (6). If $f(\cdot)$ is strictly log-concave, any solution $\mathbf{v}^*$ to (7) solves (6), and prizes v_r^* are strictly decreasing in r as long as they are positive. In either case, $v_r^* \in q(\beta_r, \lambda^*; e^*)$ for some $\lambda^* > 0$.*

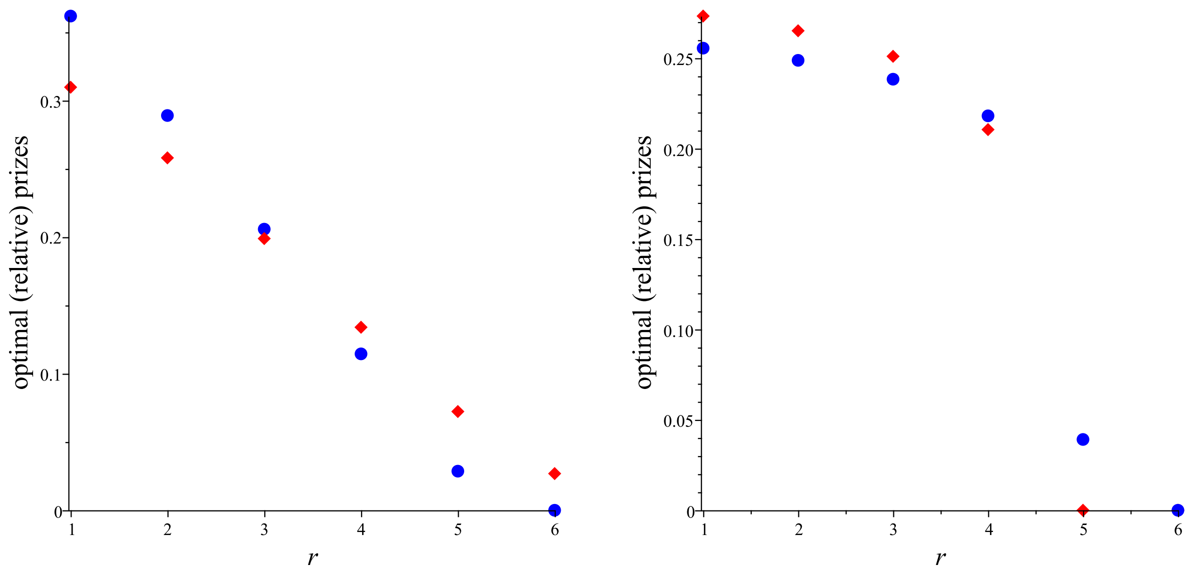

Figure 1: Optimal prize allocation for Gumbel distribution with parameter 1 (its pdf is $f(x) = \exp[-x - \exp(-x)]$), $n = 6$ and budget equal to 10. *Left:* $U(v, e; \alpha) = \log(\alpha(v - e^2) + 1)$, $\alpha = 1$ (and $e^* \approx 0.398$) (blue circles) and $\alpha = 5$ (and $e^* \approx 0.298$) (red diamonds). *Right:* $U(v, e) = \frac{1-\exp(-\alpha(v-e^2))}{1-\exp(-\alpha)} - (18\alpha - 44)e^2$, $\alpha = 2.5$ (and $e^* \approx 0.091$) (blue circles) and $\alpha = 3$ (and $e^* \approx 0.015$) (red diamonds).

Proposition 2 can be understood as follows. The objective in (10) satisfies strictly increasing differences in (β, v), and hence the monotone comparative statics imply that correspondence $q(\beta, \lambda; e^*)$ is increasing in β. Therefore, if coefficients β_r are strictly decreasing in r—as is the case when $f(\cdot)$ is strictly log-concave, see Lemma 1(ii)—then v_r^* is decreasing in r for any solution $\mathbf{v}^*$ to (7), even if the objective in (10) is not concave and the set of solutions is multi-valued and non-convex. Thus, $\mathbf{v}^*$ also solves problem (6).

Figure 1 shows examples of optimal prize schedules characterized by Proposition 2. For illustration, we use the Gumbel distribution of noise whose pdf is strictly log-concave. Interestingly, it may be optimal to give positive prizes to *all* ranks, including the very last, even though $\beta_n < 0$ (see Lemma 1(i)), as is the case in Figure 1(left). It is easy to see from (10) that $v_n = 0$ is always optimal in the separable case when $U_{ve} = 0$. With nonseparability, if $U_{ve} > 0$ is large enough, incentivizing the lowest ranks helps smooth out the marginal cost of effort.

3.2 Optimal prizes with the monotonicity constraint

As discussed in the previous section, when β_r are decreasing, the monotonicity constraint is not binding. This is no longer the case if β_r is strictly increasing for some r. Unrestricted optimal prizes then become nonmonotone; hence, they need to be appropriately "ironed". Generically, optimal prizes thus have a step-wise decreasing structure, with a sequence of *critical points* $1 \leq r_1 < \ldots < r_K \leq n$ such that v_r is strictly decreasing in r at the critical points, as long as prizes are positive, and remains constant between them: $v_1 = \ldots = v_{r_1} > v_{r_1+1} = \ldots = v_{r_2} > v_{r_2+1}$, etc.[13]

Let $\bar{\beta}_{r:r'} = \frac{1}{r'-r+1} \sum_{k=r}^{r'} \beta_k$ denote the *partial average* of coefficients β_r between ranks $r \leq r'$. Further, set $r_0 = 0$ and define a sequence of critical points $r_1, \ldots, r_K$, where the number of points, K, is at least 1 and at most n, recursively as follows:

$$r_{k+1} = \max\{r : \bar{\beta}_{r_k+1:l} \leq \bar{\beta}_{r_k+1:r} \ \forall l = r_k+1, \ldots, r\}. \tag{11}$$

Thus, r_1 is defined as the largest r such that the average $\bar{\beta}_{1:r}$ is increasing, then r_2 is defined as the largest r such that the average $\bar{\beta}_{r_1+1:r}$ is increasing, etc. By construction, $\bar{\beta}_{r_{k-1}+1:r_k}$ is strictly decreasing in k. The following proposition characterizes the optimal prize structure.

Proposition 3 *There exists a solution $(\mathbf{v}^*, e^*)$ to problem (5) with the following structure:*

(i) $v^*_{r_{k-1}+1} = \ldots = v_{r^*_k}$ for each $k = 1, \ldots, K$, where $0 = r_0 < r_1 \leq \ldots \leq r_K \leq n$ is the sequence of critical points defined above.

(ii) $v^*_{r_k}$ are strictly decreasing in k for $k = 1, \ldots, s$ and $v^*_r = 0$ for $r \geq r_s + 1$ for some $s \leq K$.

(iii) $v^*_{r_k} \in q(\bar{\beta}_{r_{k-1}+1:r_k}, \lambda^*; e^*)$ for some $\lambda^* > 0$.

The location of critical points, and hence the location of (potential) positive prize differentials, is determined only by coefficients β_r, that is, only by the distribution of noise. If $v^*_{r_s} > v^*_{r_{s+1}} = 0$ for some s, then $v^*_r = 0$ for all $r \geq r_s + 1$, and there are no more positive prize differentials. The location of such s—the number of distinct positive prizes—is determined jointly by the properties of noise and the utility function U.

Thus, we have shown that, in the general case, (potentially) distinct optimal prizes $v^*_{r_k}$ behave in essentially the same way as optimal prizes in Section 3.1, with partial averages $\bar{\beta}_{r_{k-1}+1:r_k}$ playing the role of monotonically decreasing coefficients β_r. Indeed,

[13]For the purposes of this discussion, it is convenient to introduce a fictitious prize $v_{n+1} = 0$.

when β_r are not monotonically decreasing, the optimal prizes—which tend to follow their behavior, see intuition after Proposition 2—have to be "ironed" to satisfy the monotonicity constraint and the same prize is given to several adjacent ranks. This prize has to be then optimal for these ranks on average, that is, it solves the average problem (10) for them. Hence, coefficients β_r are replaced by their partial averages $\bar{\beta}_{r_{k-1}+1:r_k}$ and a non-monotonic sequence of β_r is "ironed" into a monotonically decreasing sequence of $\bar{\beta}_{r_{k-1}+1:r_k}$.

3.3 Optimal prize schedules for some classes of distributions

Proposition 2 characterizes the optimal prize schedule when the noise distribution is log-concave. This section provides illustrations—corollaries—of the key Proposition 3 for other classes of noise distributions. We start with the heavy-tailed distributions (Section 3.3.1) and then consider unimodal distributions under an additional restriction on the utility function (Section 3.3.2).

3.3.1 Heavy-tailed distributions

The first critical point in the optimal allocation of prizes, r_1, is defined as the largest r that maximizes $\bar{\beta}_{1:r} = \frac{1}{r}\sum_{k=1}^{r} \beta_k$ – the running average of coefficients β_r. As shown by Drugov and Ryvkin (2020), this running average can be written in the form

$$\bar{\beta}_{1:r} = \frac{1}{n}\mathrm{E}(h(X_{(n-r:n)})). \tag{12}$$

Here, $h(x) = \frac{f(x)}{1-F(x)}$ is the failure (or hazard) rate of noise, and $X_{(n-r:n)}$ is its order statistic. The role of the failure rate can be understood intuitively from the following arguments. Eq. (12) can be rewritten as $\bar{\beta}_{1:r} = \frac{1}{n}\int f(x|X \geq x)f_{(n-r:n)}(x)dx$, where the failure rate $\frac{f(x)}{1-F(x)}$ is written as the density at x of variable X conditional on $X \geq x$. Thus, $\bar{\beta}_{1:r}$ is determined by the density at zero of the difference between X and $X_{(n-r:n)}$ conditional on $X \geq X_{(n-r:n)}$. Indeed, the probability of reaching a rank of at least r can be expressed as the probability of surpassing the r-th highest noise realizations out of n conditional on X being among the top r realizations, multiplied by the probability that X is in the top r (equal $\frac{r}{n}$).

Representation (12) together with Proposition 3 immediately imply $r_1 = n-1$ if noise has a decreasing failure rate (DFR). This leads to maximum prize sharing.

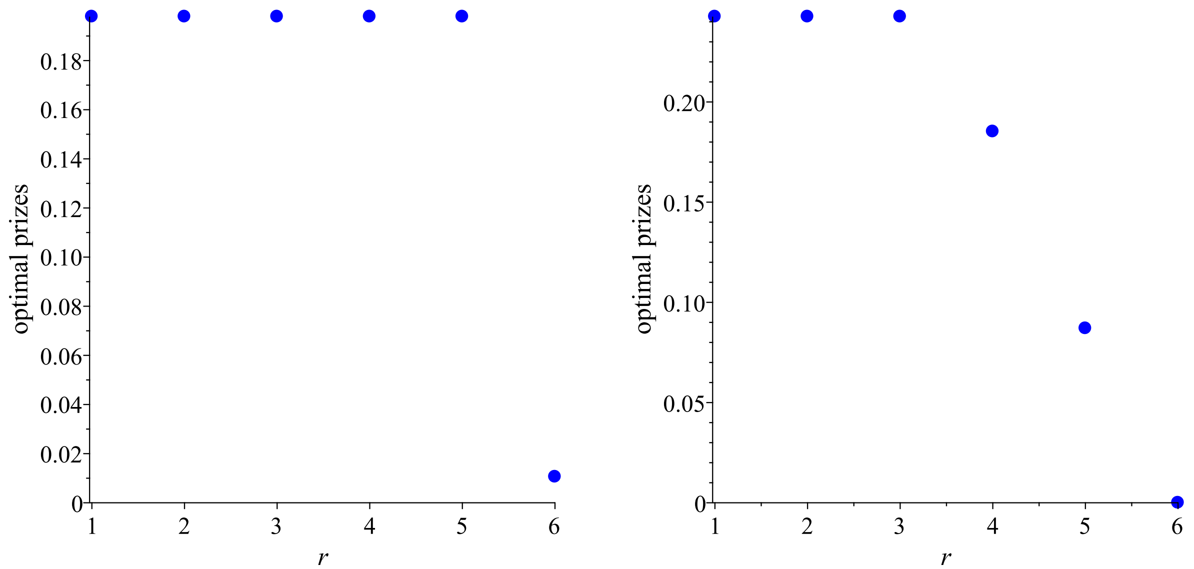

Figure 2: Optimal prize allocation for $U(v, e) = \log(7(v - 0.05e^2) + 1)$, budget equal to 1 and $n = 6$. *Left:* Pareto distribution with parameters $(1, 1)$ (its pdf is $f(x) = \frac{1}{x^2}\mathbb{1}_{x\geq 1}$). *Right:* Burr distribution with parameters $(2, 1)$ (its pdf is $f(x) = \frac{2x}{(x^2+1)^2}\mathbb{1}_{x\geq 0}$).

Corollary 1 *If $f(\cdot)$ is DFR, then the following allocation of prizes is optimal:*

$$v_r^* = \frac{1 - v_n}{n-1} > v_n, \quad r = 1, \ldots, n-1. \tag{13}$$

Allocation (13) can be characterized as the "extreme punishment" tournament. It is the polar opposite of WTA in that it punishes the worst-performing agent instead of rewarding the top performer. This allocation is optimal for DFR distributions for any utility function and hence, even when agents are risk-neutral. It is illustrated in Figure 2(left) for Pareto distribution.

A more general class of distributions that can be characterized as heavy-tailed are those having a unimodal failure rate (first IFR, then DFR). Yet, Lemma 1(iv) requires a slightly stronger condition for the unimodality of β_r which simplifies the sequence of critical points $1 \leq r_1 < \ldots < r_K$ in Proposition 3 leading to the following corollary.

Corollary 2 *If $f(\cdot)$ is first log-concave and then log-convex with $f(\overline{x}) = 0$, then it is optimal to assign r_1 equal prizes at the top, $v_1^* = \ldots = v_{r_1}^* \in q(\bar{\beta}_{1:r_1}, \lambda^*; e^*)$, followed by decreasing prizes $v_r^* \in q(\beta_r, \lambda^*; e^*)$ for $r = r_1 + 1, \ldots, n$, possibly with some zero prizes at the end.*

Indeed, denoting by r_m the largest r such that $\beta_r \geq \beta_{r-1}$, it is clear that $r_1 \geq r_m$—the maximum of $\bar{\beta}_{1:r}$ is to the right of the mode of β_r—implying that β_r is decreasing for $r > r_1$. Therefore, a prize schedule that is flat for $r \leq r_1$ and decreasing for $r > r_1$ is optimal. Examples of such distributions include the log-normal distribution, the Burr distribution, and F- and Beta-distributions for some parameters (see, e.g., Bagnoli and Bergstrom, 2005). Figure 2(right) shows an example with the Burr distribution which has pdf $f(x) = \frac{2x}{(x^2+1)^2}\mathbb{1}_{x \geq 0}$.

3.3.2 Unimodal distributions

As we mentioned after Proposition 2, it might be optimal to give a positive prize for a rank r with a negative β_r. This also means that the number of strictly positive prizes can be up to n as in the examples in Figures 1(left) and 2(left). However, when the noise distribution is unimodal, prizes for negative β_r can be easily ruled out. This is the next proposition.

Proposition 4 *If $f(\cdot)$ is unimodal and $U_{ve}(v,e) \leq 0$ or $U_{vve}(v,e) \geq 0$, then $v_r = 0$ for any r such that $\beta_r \leq 0$ is optimal.*

When $f(\cdot)$ is unimodal, Lemma 1(v) shows that β_r is single crossing, i.e., there exists an $\hat{r} \leq n-1$ such that $\beta_r > 0$ for $r \leq \hat{r}$ and $\beta_r \leq 0$ for $r > \hat{r}$. Then, $\hat{r}$ is a critical point, as defined in (11), since all next β_r are negative and hence, the partial average necessarily decreases. All next partial averages are then negative. From (9) it is clear that all prizes in such cases should be set to zero if $U_{ve}(v,e) \leq 0$. The sufficiency of $U_{vve}(v,e) \geq 0$ is more subtle and is proven by redistributing prizes from ranks with a negative β_r to rank $r = 1$. Conditions $U_{ve}(v,e) \leq 0$ and $U_{vve}(v,e) \geq 0$ hold, for example, when $U(v,e) = k(e)u(v) - c(e)$ with $k' \leq 0$, $u' > 0$ and $u'' \leq 0$. See Figures 3 and 4 for examples.

Proposition 4 implies that positive prizes are given at most up to $\hat{r}$. Moreover, the maximum number of positive prizes $\hat{r}$ does not depend on the utility function (provided $U_{ve}(v,e) \leq 0$ or $U_{vve}(v,e) \geq 0$) but only on the noise distribution. This allows to obtain a general result on the optimality of the winner-take-all (WTA) tournament.

Corollary 3 *If $f(\cdot)$ is increasing and $U_{ve}(v,e) \leq 0$ or $U_{vve}(v,e) \geq 0$, then WTA is optimal.*

Corollary 3 obtains by showing that $\beta_r \leq 0$ for all $r > 1$ when $f(\cdot)$ is increasing.

In a separable utility case $U(v,e) = u(v) - c(e)$, Krishna and Morgan (1998) derive the "WTA principle for small tournaments", that is, the optimality of WTA when $n \leq 3$ and the noise distribution is unimodal and symmetric. Next corollary generalizes this result.

Corollary 4 *If $n \leq 3$, $f(\cdot)$ is symmetric and $U_{ve}(v,e) \leq 0$ or $U_{vve}(v,e) \geq 0$, then WTA is optimal.*

Note that the unimodality of $f(\cdot)$ is not required. Indeed, Lemma 1(i) implies for $n = 3$ that $\beta_2 = 0$ and $\beta_3 < 0$ meaning that the sequence β_r has the single-crossing structure as in the unimodal case, with $\hat{r} = 1$.[14]

4 Risk aversion and prize sharing

It is generally understood that risk aversion creates risk sharing considerations which call for more balanced prize schemes (Krishna and Morgan, 1998; Kalra and Shi, 2001). Yet, Figure 1(right) shows the opposite pattern: The prize schedule under higher risk aversion, $\alpha = 3$, is steeper than under lower risk aversion, $\alpha = 2.5$. The reason is that parameter α not only increases risk aversion but also the cost component, and the two effects may go in the same or in the opposite directions.[15] In the separable case the second effect does not matter (see Section 5) but in the nonseparable case it does. Thus, the effect of risk aversion under nonseparability is ambiguous in general.

In this section we study how the optimal prize structure is affected by transformations of utility function U into $\tilde{U}$, where $\tilde{U}$ is, in some sense, more risk averse than U. We are interested, in particular, in conditions under which higher risk aversion leads to more prize sharing. We assume that $\tilde{U}(v,e)$ is regular. We use the concept of majorization (Marshall, Olkin and Arnold, 2011) to rank prize schedules by the degree of prize sharing as is done in, e.g., Vojnović (2016) and Fang, Noe and Strack (2020).

Definition 1 *For two vectors $\mathbf{v}, \tilde{\mathbf{v}} \in \mathbb{R}^n_+$ whose components are arranged in descending order and $\sum_{k=1}^n v_k = \sum_{k=1}^n \tilde{v}_k$, $\mathbf{v}$ majorizes $\tilde{\mathbf{v}}$ if $\sum_{k=1}^r v_k \geq \sum_{k=1}^r \tilde{v}_k$ for all $r = 1, \ldots, n$.*

[14] Another consequence of a unimodal $f(\cdot)$ is that the requirement in Corollary 2 that $f(\cdot)$ is first log-concave and then log-convex with $f(\overline{x}) = 0$ can be relaxed because β_r only needs to be unimodal for $r \leq \hat{r}$. For example, some symmetric heavy-tailed distributions, such as the t-distribution, are log-concave in an interval around zero and log-convex otherwise. It can then be shown based on Lemma 1 that the upper half of β_r are positive and their sequence is unimodal. In this case, prize structure as in Corollary 2 is still optimal.

[15] This example is discussed in more detail at the end of this section.

Components of a prize schedule $\mathbf{v}$ such that the budget constraint holds can be interpreted as a probability mass function (pmf) of a discrete random variable taking values $1, \ldots, n$. Definition 1 then produces the inequality between the corresponding cumulative mass functions (cmfs) stating that the second random variable is larger than the first one in the FOSD sense, i.e., the mass in $\tilde{\mathbf{v}}$ is shifted to the right relative to $\mathbf{v}$. It is, therefore, natural to interpret $\tilde{\mathbf{v}}$ as involving more prize sharing than $\mathbf{v}$ if $\mathbf{v}$ majorizes $\tilde{\mathbf{v}}$. Graphically, if $\mathbf{v}$ majorizes $\tilde{\mathbf{v}}$ then $\mathbf{v}$ crosses $\tilde{\mathbf{v}}$ from above, as in Figure 1.

4.1 Risk aversion for nonseparable utility functions

The following function plays a key role in our results:

$$\gamma(v, e) = \frac{U_{ve}(v, e)}{U_v(v, e)}. \tag{14}$$

We will refer to $\gamma(v, e)$ as the *coefficient of relative complementarity* (or, for brevity, complementarity) between money and effort. It measures the relative sensitivity of the marginal utility of money to changes in effort. For a separable utility function, $\gamma = 0$. When $U(v, e) = \eta(v - c(e))$, with $\eta(\cdot)$ concave, $\gamma \geq 0$. For an example of $\gamma \leq 0$, consider $U(v, e) = k(e)u(v) - c(e)$, with $k(\cdot)$ decreasing.

The standard definition of higher risk aversion—$\tilde{U}$ is more risk averse than U if $\tilde{U} = \phi(U)$, where $\phi(\cdot)$ is strictly increasing and concave—is also used for functions with multiple arguments (see Kihlstrom and Mirman, 1974, 1981). However, in the latter case it restricts the two utility functions to represent the same ordinal preferences. Here we use a weaker definition.

Definition 2 *$\tilde{U}$ is more risk averse than U if, for all (v, e),*
(a) $-\frac{\tilde{U}_{vv}(v,e)}{\tilde{U}_v(v,e)} \geq -\frac{U_{vv}(v,e)}{U_v(v,e)}$, and
(b) $\tilde{\gamma}(v, e) \geq \gamma(v, e)$.

Part (a) of Definition 2 is based on the ranking of the standard coefficient of absolute risk aversion, which may depend on effort. It is equivalent to $\frac{U_v(v',e)}{U_v(v,e)} \geq \frac{\tilde{U}_v(v',e)}{\tilde{U}_v(v,e)}$ for any $v' > v$ and e. Part (b) states that the marginal utility of money is more sensitive to effort under $\tilde{U}$ than under U.[16] There is no need for a condition involving U_{ee} since there is no uncertainty about the effort. For separable utilities, $U(v, e) = u(v) - c(e)$ and $\tilde{U}(v, e) = \tilde{u}(v) - \tilde{c}(e)$, Definition 2 is equivalent to $\tilde{u}$ being more concave than u; whereas

[16] This interpretation holds assuming $\gamma, \tilde{\gamma} \geq 0$.

in the simplest nonseparable case, $U(v,e) = \eta(v - c(e))$ and $\tilde{U}(v,e) = \tilde{\eta}(v - c(e))$, it is equivalent to $\tilde{\eta}$ being more concave than η.

4.2 Compensated transformations of utility functions

For a given prize schedule $\mathbf{v}$, the equilibrium effort $\hat{e}$ is a solution to the first-order condition (3) that can be written in the form

$$\sum_{r=1}^{n} \beta_r U(v_r, e) = -\frac{1}{n} \sum_{r=1}^{n} U_e(v_r, e). \tag{15}$$

The left-hand side of (15) represents the marginal benefit of effort, whereas the right-hand side is the marginal cost. In the nonseparable case, both of these are affected by prizes; therefore, a general transformation of utility will affect the optimal prize structure not only directly, via a change in the curvature of U with respect to v, but also indirectly via a change in $\hat{e}$. The latter effect can be arbitrary; therefore, in order to isolate the effect of risk aversion, we formulate the main result of this section—Proposition 5—under the assumption that the optimal effort does not change. That is, we consider *compensated* transformations of utility $U \to \tilde{U}$ such that the implemented equilibrium effort is the same under the corresponding optimal prize structures.[17] A result for uncompensated transformations of utility where the equilibrium effort can change is provided in Proposition 6, under additional restrictions.

Compensated utility transformations can be understood similar to the concept of compensated price changes giving rise to compensated demand in classic consumer theory. There, a change in prices leads to a change in the optimal consumption bundle. Due to the income effect, this change can be arbitrary, and universal comparative statics of demand with respect to price transformations cannot be established. If, however, the income effect is removed by adjusting the consumer's budget in a way that preserves the consumer's utility level, the resulting compensated demand admits universal comparative statics. In our case, the implemented optimal equilibrium effort e^* plays the role of the principal's utility. For an arbitrary transformation $U \to \tilde{U}$, e^* changes in an unrestricted way and, due to the direct effect of e^* on optimal prizes, comparative statics for the latter are also unrestricted. In contrast, universal comparative statics generalizing known results for separable utility can be established for compensated utility transformations.

[17] In the separable case, this restriction is immaterial because the optimal prize structure is independent of effort.

Operationally, a compensated utility transformation can be implemented by adjusting the prize budget. Suppose, without loss, that the original budget is $b = 1$, the assumption we keep throughout this section.[18] Prizes $\mathbf{v}$ are then interpreted as *relative prizes*. Lemma B1 in Appendix B shows that, under a relatively mild condition, a budget adjustment b^c exists such that the optimal effort does not change when the utility function is transformed. Hence, for a given utility transformation $U \to \tilde{U}$, compensated utility can be defined as $\tilde{U}^c(v, e) = \tilde{U}(b^c v, e)$ with unit budget.[19]

4.3 Compensated effects of risk aversion

We are now ready to state the main result of this section.

Proposition 5 *Consider two utility functions, U and $\tilde{U}$, let $\mathbf{v}^*$ and $\tilde{\mathbf{v}}$ denote the corresponding optimal prize schedules, and suppose that the implemented equilibrium effort is the same. If (a) $\tilde{U}$ is more risk averse than U, and (b) the corresponding coefficients of relative complementarity satisfy $\tilde{\gamma}_v(v, e) \leq \gamma_v(v, e) \leq 0$, then*

(i) $\tilde{\mathbf{v}}$ has more positive prizes than $\mathbf{v}^*$;

(ii) $\mathbf{v}^*$ majorizes $\tilde{\mathbf{v}}$.

Proposition 5 extends the result that a higher risk aversion leads to more prize sharing. Also, even though part (i)—about the number of prizes—follows from part (ii) it is stated for explicitness since the number of prizes is often discussed the literature, particularly, in WTA vs. non-WTA schemes as in, e.g., Krishna and Morgan (1998), Kalra and Shi (2001) and Fu, Wang and Wu (2021). When utility is separable, condition (b) is trivially satisfied since $\tilde{\gamma}(v, e) = \gamma(v, e) = 0$. Moreover, the equilibrium effort can change in this case (see Section 5 for details). Intuitively, concavity in the utility of money pushes optimal prize schedules in the direction of prize sharing.

The nonseparability of money and effort in the utility function introduces additional considerations, due to the dependence of the marginal cost of effort—the right-hand side of (15)—on prizes. The relative strength of this effect, for an agent ranked r, is measured by $\gamma(v_r, e) = \frac{U_{ve}(v_r, e)}{U_v(v_r, e)}$, and the total marginal effect of prize v_r on the marginal utility of

[18] For $b \neq 1$, we can redefine prizes and utility as $\mathbf{v} \to \frac{1}{b}\mathbf{v}$ and $U(v, e) \to U(bv, e)$, and reduce problems (5) and (6) to the corresponding problems with unit budget.

[19] Another way to produce a compensated utility transformation is to add a term to the utility function that depends only on the effort, as is done in Figure 1(right). The results of this section are not affected as long as partial derivatives of various orders of $U(v, e)$ with respect to v as well as cross derivatives do not change.

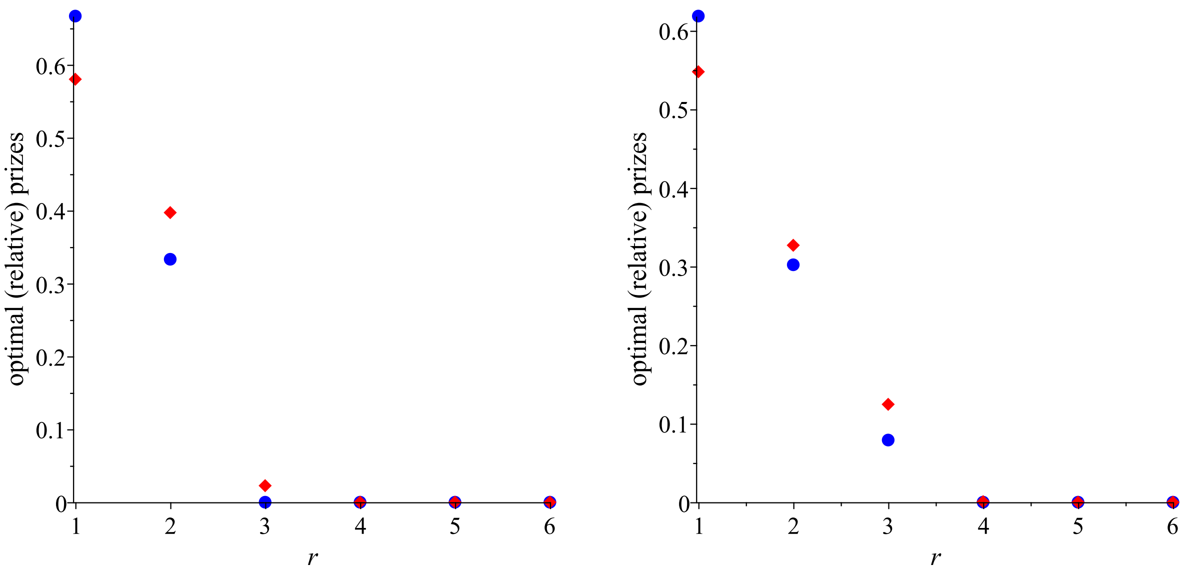

Figure 3: Optimal prize allocation for Gumbel distribution with parameter 1 (its pdf is $f(x) = \exp[-x - \exp(-x)]$) and $n = 6$ (in which case $\beta_r < 0$ for $r \geq 5$). *Left*: $U(v, e; \alpha) = (1 - e^2)\frac{1-\exp(-\alpha v)}{1-\exp(-\alpha)} - e^2$ with $\alpha = 1$ (and budget equal to 1) (blue circles) and $\alpha = 3$ (and budget equal to 0.6079) (red diamonds). In both cases $e^* \approx 0.0637$. *Right*: $U(v, e; \alpha) = (1 - 0.1e^2)\frac{v^{1-\alpha}}{1-\alpha} - e^2$ with $\alpha = 0.4$ (and budget equal to 1.25) (blue circles) and $\alpha = 0.55545$ (and budget equal to 0.5) (red diamonds). In both cases $e^* \approx 0.154$.

effort can be written as $A_{v_r}(\mathbf{v}, e) = [\beta_r + \frac{1}{n}\gamma(v_r, e)]U_v(v_r, e)$. The first part of condition (b) in Proposition 5, $\gamma_v(v, e) \leq 0$, therefore, ensures that A_{v_r} is decreasing in v_r, or at least it is not increasing too fast if $\beta_r < 0$. The second part of condition (b), $\tilde{\gamma}_v(v, e) \leq \gamma_v(v, e)$, together with the requirement that $\tilde{U}$ is more risk averse than U, implies that the marginal utility of effort is more sensitive to prizes under $\tilde{U}$ than under U, especially for prizes at the bottom where both $\gamma(v_r, e)$ and $\tilde{\gamma}(v_r, e)$ are the largest. This leads to larger prizes at the bottom being optimal under $\tilde{U}$, and the majorization result follows. Proposition 5 is illustrated in Fig. 3.

For a nonseparable utility function of the form $U(v, e) = \eta(v - c(e))$, condition (b) of Proposition 5 is satisfied by strictly increasing, concave transformations $\eta \to \tilde{\eta} = \phi(\eta)$ such that $\phi(\cdot)$ has the non-increasing absolute risk aversion (NIARA) property: $-\frac{\phi''(x)}{\phi'(x)}$ is decreasing in x. The following lemma generalizes this observation.

Lemma 2 *Suppose that $U \to \tilde{U} = \phi(U)$, where $\phi(\cdot)$ is strictly increasing, concave and satisfies NIARA, and $U_{ve} \geq 0$ Then, $\tilde{\gamma}_v(v, e) \leq \gamma_v(v, e)$.*

Thus, if the original utility U has $\gamma(v, e) \geq 0$ and $\gamma_v(v, e) \leq 0$, and it is transformed by a function $\phi(\cdot)$ with the properties as in Lemma 2, Proposition 5 applies. For $U(v,e) = \eta(v - c(e))$, $\gamma(v,e) \geq 0$ always holds, and the property $\gamma_v(v,e) \leq 0$ is equivalent to $-\frac{\eta''}{\eta'}$ decreasing, i.e., to $\eta(\cdot)$ having NIARA. Hence, starting with a NIARA utility function and then using a NIARA transformation provides sufficient conditions for Proposition 5.

Finally, in one particular case—when the parameter only multiplies the prize—the compensated effect of a change in this parameter on the optimal prize schedule is absent.

Lemma 3 *Consider a parametrized family of utility functions $U(v,e;\rho) = U(\rho v, e)$ where the prize is multiplied by parameter $\rho > 0$. Suppose the implemented equilibrium effort is fixed at e^*, and prize schedule $\mathbf{v}^1$ is optimal for $\rho = \rho_1$ under unit budget. Then prize schedule $\mathbf{v}^2 = \frac{\rho_1}{\rho_2}\mathbf{v}^1$ is optimal for $\rho = \rho_2$ under budget $b = \frac{\rho_1}{\rho_2}$.*

For example, suppose that $U(v,e) = k(e)u(v) - c(e)$, where $u(v) = 1 - \exp(-\alpha v)$ is the non-normalized CARA utility function. Then, following a change in α and the budget to keep the equilibrium effort constant, the optimal relative prize schedule does not change.

4.4 Non-compensated effects of risk aversion

While Proposition 5 holds under relatively mild assumptions, it requires that the implemented equilibrium effort stays the same when U is transformed into $\tilde{U}$. With additional restrictions imposed on $\tilde{U}$, the majorization result can also be obtained when the equilibrium effort is allowed to change, as described in the following proposition.

Proposition 6 *Consider two utility functions, U and $\tilde{U}$, and let $(\mathbf{v}^*, e^*)$ and $(\tilde{\mathbf{v}}, \tilde{e})$ denote the corresponding optimal prize schedules and implemented equilibrium efforts. If (a) $\tilde{U}$ is more risk averse than U, (b) $\tilde{\gamma}_v(v,e) \leq \gamma_v(v,e) \leq 0$, (c) $\frac{\tilde{U}_{vee}(v,e)}{\tilde{U}_{ve}(v,e)}$ is decreasing in v, (d) $\tilde{v}_{r_k} = 0$ whenever $\bar{\beta}_{r_{k-1}+1:r_k} < 0$, and (e) $\tilde{e} \geq e^*$, then*

(i) $\tilde{\mathbf{v}}$ has more positive prizes than $\mathbf{v}^*$;

(ii) $\mathbf{v}^*$ majorizes $\tilde{\mathbf{v}}$.

Requiring the equilibrium effort not to decrease when U is transformed into $\tilde{U}$ to control the direction of its change, Proposition 6 imposes two additional restrictions relative to Proposition 5—conditions (c) and (d). To interpret condition (c), consider nonseparable utilities $\tilde{U}(v,e) = \tilde{\eta}(v - \tilde{c}(e))$, for which it takes the form of non-increasing absolute prudence (NIAP), $\frac{\partial}{\partial v}\left(-\frac{\tilde{\eta}'''}{\tilde{\eta}''}\right) \leq 0$. Kimball (1993) argues that NIAP is a natural property and shows that a combination of NIARA and NIAP is equivalent to "standard risk

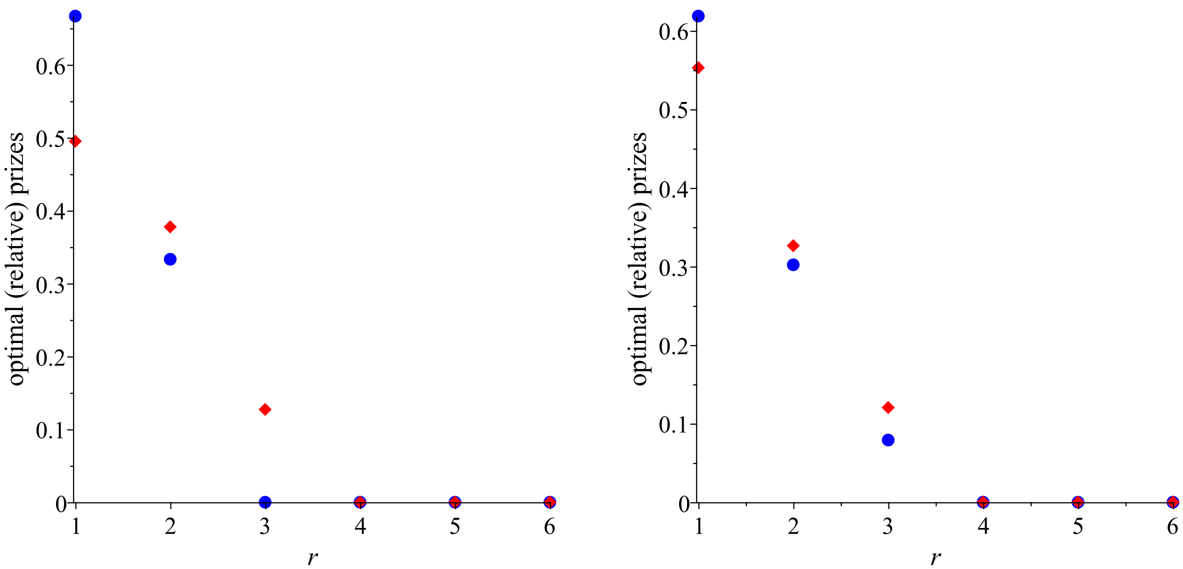

Figure 4: Optimal prize allocation for Gumbel distribution with parameter 1 (its pdf is $f(x) = \exp[-x - \exp(-x)]$), $n = 6$ (in which case $\beta_r < 0$ for $r \geq 5$). *Left:* $U(v, e; \alpha) = (1 - e^2)\frac{1-\exp(-\alpha v)}{1-\exp(-\alpha)} - e^2$ and budget equal to 1, $\alpha = 1$ (and $e^* \approx 0.0637$) (blue circles) and $\alpha = 3$ (and $e^* \approx 0.0796$) (red diamonds). *Right:* $U(v, e; \alpha) = (1 - 0.1e^2)\frac{v^{1-\alpha}}{1-\alpha} - e^2$ and budget equal to 1.25, $\alpha = 0.4$ (and $e^* \approx 0.154$) (blue circles) and $\alpha = 0.55545$ (and $e^* \approx 0.553$) (red diamonds).

aversion"—facing two independent lotteries is worse than facing one of them. Proposition 6 is illustrated in Fig. 4.

Condition (d) requires that positive prizes are given only for ranks $r \leq r_k^*$ such that $\bar{\beta}_{r_{k-1}^*+1:r_k^*} \geq 0$. Proposition 4 shows that this is the case, for instance, when $f(\cdot)$ is unimodal and $U_{ve}(v, e) \leq 0$ or $U_{vve}(v, e) \geq 0$. These conditions are satisfied by the example in Fig. 4.

The example in Figure 1(right) with utility function $U(v, e) = \frac{1-\exp(-\alpha(v-e^2))}{1-\exp(-\alpha)} - (18\alpha - 44)e^2$—in which higher risk aversion reduces prize sharing—does not satisfy conditions (d) and (e) of Proposition 6. Indeed, equilibrium effort decreases with α and a positive prize is given for $r = 5$ even though $\beta_r < 0$ for $r \geq 5$. Conditions (b) and (c) are satisfied since both $\gamma(v, e) = 2\alpha e$ and $\frac{U_{vee}(v,e)}{U_{ve}(v,e)} = 2\alpha e + \frac{1}{e}$ are constant in v. Intuitively, α increases risk aversion via the first term in $U(v, e)$ and makes the effort more costly via the second term. The second effect makes optimal effort decrease with α and—since $U_{vee}(v, e) > 0$—this decreases the complementarity between v and e, $U_{ve}(v, e)$. The complementarity is positive, $U_{ve}(v, e) > 0$ and hence, it is a force towards spreading the prizes to decrease the marginal cost of effort—which also explains why a prize is given for a rank with a negative β_r, cf. Proposition 4. This reduced complementarity effect goes against the risk aversion effect and—for the parameter values used in the example—dominates.

5 Separable utility functions

The existing literature studying the impact of risk aversion on the optimal prize schedule in rank-order tournaments considers the separable utility function $U(v, e) = u(v) - c(e)$, and some particular settings: Krishna and Morgan (1998) consider tournaments of up to four players while Kalra and Shi (2001) consider only logistic and uniform noise distributions. In this section we obtain a generalization of these previous results as corollaries of our general results of Sections 3 and 4.

The key simplification brought about by the separability of the utility function is a dissociation of the equilibrium effort and optimal prizes. Indeed, the first-order condition (3) becomes

$$\sum_{r=1}^{n} \beta_r u(v_r) = c'(\hat{e}) \tag{16}$$

and the principal's problem (5) then writes

$$\max_{\mathbf{v}} \sum_{r=1}^{n} \beta_r u(v_r) \quad \text{s.t. } \mathbf{v} \in \mathcal{V}_b. \tag{17}$$

Finally, the correspondence $q(\beta, \lambda; e^*)$ in (10) becomes

$$q^{sep}(\beta, \lambda) = \begin{cases} 0, & \text{if } \beta u'(0) \leq \lambda \\ 1, & \text{if } \beta u'(1) \geq \lambda \\ \{v : \beta u'(v) = \lambda\}, & \text{otherwise,} \end{cases} \tag{18}$$

The optimal prize schedule—the solution to problem (17)—is characterized in the next corollary to Proposition 3.

Corollary 5 *When $U(v, e) = u(v) - c(e)$, the optimal prize schedule $\mathbf{v}^*$ has the following structure:*

(i) $v^*_{r_{k-1}+1} = \ldots = v_{r^*_k}$ for each $k = 1, \ldots, K$, where $0 = r_0 < r_1 \leq \ldots \leq r_K \leq n$ is the sequence of critical points as in Proposition 3.

(ii) $v^*_{r_k}$ are strictly decreasing in k for $k = 1, \ldots, s$ and $v^*_r = 0$ for $r \geq r_s + 1$ for some $s \leq K$.

(iii) $v^*_{r_k} = q^{sep}(\bar{\beta}_{r_{k-1}+1:r_k}, \lambda^*)$*, where* λ^* *is the unique solution to*

$$\sum_{k=1}^{K}(r^*_k - r^*_{k-1})q^{sep}(\bar{\beta}_{r^*_{k-1}+1:r^*_k}, \lambda) = 1. \tag{19}$$

Hence, perhaps surprisingly, the structure of the optimal prizes is the same as in the general case. Indeed, the critical points $r_1, \dots, r_K$ are determined by the noise distribution alone and do not depend on the utility function. This also implies that the "ironing" necessary to satisfy the monotonicity constraint is the same. The only difference with the general case comes in (iii) since now the optimal prizes are found independently of the equilibrium effort.

The separable form of the utility function greatly simplifies the analysis of the effect of the risk aversion which is the next proposition.

Proposition 7 *Suppose* $U(v, e) = u(v) - c(e)$*. Consider two utility functions* $u, \tilde{u} : [0, 1] \to [0, 1]$ *and let* $\mathbf{v}^*$ *and* $\tilde{\mathbf{v}}$ *denote the corresponding optimal prize schedules. If* $\tilde{u}(\cdot)$ *is more risk-averse than* $u(\cdot)$*, then*

(i) $\tilde{\mathbf{v}}$ *has more positive prizes than* $\mathbf{v}^*$*;*

(ii) $\mathbf{v}^*$ *majorizes* $\tilde{\mathbf{v}}$*.*

In other words, in the separable case higher risk aversion *always* leads to more prize sharing. Formally, this result is not a direct corollary of Propositions 5 and 6 since they both restrict the equilibrium effort but its proof is very similar (and simpler).

Proposition 7 can be used to analyze the consequences of a higher budget. While it is obvious that the equilibrium effort goes up the effect on the optimal prize schedule is less clear. Fu, Wang and Wu (2021) find that a higher budget makes WTA less likely to be optimal, under certain conditions. The next corollary provides a general answer.

Corollary 6 *Suppose* $U(v, e) = u(v) - c(e)$ *and the budget of the contest designer is* b*. If* $u(\cdot)$ *has increasing relative risk aversion then a higher budget increases prize sharing, that is,* $\mathbf{v}''/b''$ *is majorized by* $\mathbf{v}'/b'$ *for* $b'' > b'$*. If* $u(\cdot)$ *is CRRA, then* $\mathbf{v}'/b' = \mathbf{v}''/b''$*.*

If $u(\cdot)$ has decreasing relative risk aversion then $\mathbf{v}''/b''$ majorizes $\mathbf{v}'/b'$ for $b'' > b'$, that is, a higher budget leads to less prize sharing.

In case of CARA and CRRA utility functions and a log-concave noise distribution the optimal prize schedule can be obtained in a closed form.

CARA utility Suppose $f(\cdot)$ is log-concave in which case β_r is decreasing, see Lemma 1(ii). Consider a CARA utility function $u(v) = \frac{1-\exp(-\alpha v)}{1-\exp(-\alpha)}$, where $\alpha > 0$ is the constant absolute risk aversion parameter. This gives $u'(v) = \frac{\alpha \exp(-\alpha v)}{1-\exp(-\alpha)}$ and the correspondence $q^{sep}(\beta, \lambda)$ in (18) becomes

$$q^{sep}(\beta, \lambda) = \begin{cases} 0, & \text{if } \frac{\beta\alpha}{1-\exp(-\alpha)} \leq \lambda \\ 1, & \text{if } \frac{\beta\alpha \exp(-\alpha)}{1-\exp(-\alpha)} \geq \lambda \\ \frac{1}{\alpha} \ln \frac{\beta\alpha}{\lambda(1-\exp(-\alpha))}, & \text{otherwise} \end{cases}$$

The optimal allocation of prizes is given in Corollary 5. Since β_r are monotonically decreasing the equation (19) for λ^* is

$$\sum_{r=1}^{\hat{r}} q^{sep}(\beta_r, \lambda) = \frac{1}{\alpha} \sum_{r=1}^{s} \ln \frac{\beta_r \alpha}{\lambda(1-\exp(-\alpha))} = 1,$$

where $s \leq \hat{r}$ is the optimal number of positive prizes (to be determined below). Then,

$$\lambda^* = \frac{\alpha}{1-\exp(-\alpha)} \left[\frac{\prod_{r=1}^{s} \beta_r}{\exp(\alpha)} \right]^{\frac{1}{s}}.$$

and finally

$$v_r^* = \frac{1}{s} + \frac{1}{\alpha} \ln \frac{\beta_r}{\left(\prod_{k=1}^{s} \beta_k\right)^{\frac{1}{s}}}, \quad r = 1, \ldots, s; \quad v_r^* = 0, \quad r = s+1, \ldots, n.$$

The expression in parentheses is the geometric mean of coefficients β_k. Thus, v_r is above (below) $\frac{1}{s}$ if β_r is above (below) this geometric mean. The number of positive prizes, s, is defined as $s = \max \left\{ s' \leq \hat{r} : \frac{1}{s'} + \frac{1}{\alpha} \ln \frac{\beta_{s'}}{\left(\prod_{k=1}^{s'} \beta_k\right)^{\frac{1}{s'}}} > 0 \right\}$.

By definition, a higher α means higher risk aversion, and Proposition 7 applies. Since CARA utility has increasing relative risk aversion, Corollary 6 implies that a higher budget will lead a higher prize sharing.

CRRA utility Consider now utility function $u(v) = \frac{v^{1-\rho}-1}{1-\rho}$, where $\rho \in (0,1)$ is the agents' constant relative risk aversion parameter. Correspondence $q^{sep}(\beta, \lambda)$ in (18) becomes

$$q^{sep}(\beta, \lambda) = \begin{cases} 1, & \text{if } \beta \geq \lambda \\ \left(\frac{\beta}{\lambda}\right)^{\frac{1}{\rho}}, & \text{otherwise} \end{cases}$$

Since β_r are monotonically decreasing the equation (19) for λ^*,

$$\sum_{r=1}^{\hat{r}} q^{sep}(\beta_r, \lambda) = \lambda^{-\frac{1}{\rho}} \sum_{r=1}^{\hat{r}} \beta_r^{\frac{1}{\rho}} = 1,$$

produces

$$\lambda^* = \left(\sum_{r=1}^{\hat{r}} \beta_r^{\frac{1}{\rho}} \right)^{\rho}.$$

The optimal allocation of prizes is, therefore,

$$v_r^* = q^{sep}(\beta_r, \lambda^*) = \frac{\beta_r^{\frac{1}{\rho}}}{\sum_{k=1}^{\hat{r}} \beta_k^{\frac{1}{\rho}}}, \quad r = 1, \ldots, \hat{r}; \qquad v_r^* = 0, \quad r = \hat{r} + 1, \ldots, n.$$

The maximum number of positive prizes, $s = \hat{r}$, is optimal in this case because $u'(0) = +\infty$. The resulting prizes can also be rewritten as

$$v_r^* = \frac{1}{\hat{r}} \left[\frac{\beta_r}{\left(\frac{1}{\hat{r}} \sum_{k=1}^{\hat{r}} \beta_k^{\frac{1}{\rho}} \right)^{\rho}} \right]^{\frac{1}{\rho}},$$

where the expression in parentheses is the generalized mean with exponent $\frac{1}{\rho}$. Thus, prizes are above (below) $\frac{1}{\hat{r}}$ if β_r is above (below) the generalized mean of coefficients β_k.

The coefficient of absolute risk aversion is equal to $\frac{\rho}{v}$. Hence, a higher ρ means a higher absolute risk aversion, and Proposition 7 applies. For CRRA utility, Corollary 6 says that the relative prizes do not change, that is, all prizes are simply scaled by the budget.

6 Connection to Tullock contests

Under risk neutrality, it is well-known that the contest model of Tullock (1980) can be obtained as a special case of the Lazear and Rosen (1981) tournament model with the Gumbel distribution of noise (Jia, 2008; Ryvkin and Drugov, 2020; Fu and Lu, 2012). The following lemma extends those results to a class of nonseparable utility functions with risk aversion.

Lemma 4 *Consider a multi-prize Tullock contest with discriminatory power ξ and utility $U(v,e) = k(e)u(v) - c(e)$, and suppose a symmetric equilibrium exists. Then the equilibrium effort $\hat{e}$ satisfies the first order-condition (3) for utility function $\tilde{U}(v,e) = \tilde{k}(e)u(v) - \tilde{c}(e)$, where*

$$\tilde{k}(e) = \exp\left[\int_0^e \frac{tk'(t)}{k(t)}dt\right], \quad \tilde{c}(e) = \int_0^e \frac{tc'(t)\tilde{k}(t)}{k(t)}dt. \tag{20}$$

The coefficients β_r are given by $\beta_r = \frac{\xi}{n}\left[1 + \frac{r}{n(n-r)} + \sum_{k=0}^{r-1} \frac{1}{n-r+k}\right]$.

For a separable utility, $U(v,e) = u(v) - c(e)$, Eqs. (20) are further reduced to the transformation of the cost function $\tilde{c}(e) = \int_0^e tc'(t)dt$, similar to the risk-neutral case (Ryvkin and Drugov, 2020). However, such transformations do not exist for all forms of nonseparable utility. For example, for $U(v,e) = \eta(v - c(e))$, a transformation would have to simultaneously preserve $c(e)$ inside the argument of η and transform it so that $\tilde{c}(e) = \int_0^e c'(t)tdt$, which is impossible. Thus, the theory developed in this paper is not directly applicable to utility functions of this form with multiplicative noise, such as multi-prize Tullock contests considered by Fu, Wang and Wu (2021).

7 Conclusions

In this paper, we characterize, and explore the properties of, optimal prize allocations in rank-order tournaments where agents have nonseparable preferences. Surprisingly, the structure of optimal prize allocations is robust to nonseparability. How fast optimal prizes decay, whether there are subgroups of ranks receiving the same prize, or whether WTA is optimal, depends on a combination of properties of noise and the utility function.

In contrast, the effect of risk aversion on optimal prize sharing is no longer universal. Relatively strong results can only be obtained for compensated utility changes that preserve the equilibrium effort. There, a generalized notion of risk aversion can be connected to prize sharing, in the form of the majorization order, similar to the separable case. For uncompensated utility changes, additional assumptions on higher-order mixed derivatives of utility are needed for the result to carry through. One important condition that emerges is a version of non-increasing absolute prudence (NIAP).

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Appendix A: Equilibrium existence

Let $f_m = \sup\{f(x) : x \in \mathcal{X}\}$, $f'_{\max} = \sup\{f'(x) : x \in \mathcal{X}\}$ and $f'_{\min} = \inf\{f'(x) : x \in \mathcal{X}\}$ denote the tight, possibly infinite, bounds of pdf $f(\cdot)$ and its derivative $f'(\cdot)$ on $\mathcal{X}$. We impose the following restrictions on the pdf of noise.

Assumption A1 *(a) $f(\cdot)$ is uniformly bounded; that is, $f_m < \infty$.*
(b) $f'(\cdot)$ is uniformly bounded above or below or both; that is, either $f'_{\max} < \infty$ or $f'_{\min} > -\infty$ or both.

The boundedness conditions in Assumption A1 are satisfied for many widely used distributions such as Normal, Gumbel, Laplace, Cauchy, Pareto, exponential and uniform distributions. They also allow for unbounded $f'(\cdot)$ from above or below, which is the case for beta and lognormal distributions for some parameter values.

Let $\bar{e} > 0$ denote the unique level of effort such that $U(b, \bar{e}) = 0$. It is clear that any effort above $\bar{e}$ is strictly dominated by $e = 0$; therefore, we only need to consider effort in the interval $[0, \bar{e}]$.

Next, let $g(e) = \frac{1}{n}U(b, e) + \frac{n-1}{n}U(0, e)$. Since $U(v, e)$ is concave in v, $g(e)$ is the minimum of $\frac{1}{n}\sum_{r=1}^{n} U(v_r, e)$ over all feasible prize schedules. It is easy to see that $g(0) > 0$, $g(\bar{e}) < 0$, and $g'(e) < 0$ for all $e \in (0, \bar{e})$; therefore, let $\underline{e} \in (0, \bar{e})$ denote the unique level of effort such that $g(\underline{e}) = 0$. It follows that $\frac{1}{n}\sum_{r=1}^{n} U(v_r, e) \geq 0$ for any feasible prize schedule if $e \leq \underline{e}$.

Finally, note that $U_e(v, e)$ is continuous in v, and hence $\bar{\mu}(e) = \max_{v \in [0,b]} U_e(v, e)$ and $\underline{\mu}(e) = \min_{v \in [0,b]} U_e(v, e)$ are well-defined, negative numbers for each $e \in [0, \bar{e}]$, and strictly negative for $e \in (0, \bar{e}]$.

Introduce the bounds

$$\begin{aligned} D_- &= (n-1)[(U(b,0) - U(0,\bar{e}))((n-1)f_m^2 + \max\{-f'_{\min}, 0\}) - 2\underline{\mu}(\bar{e})f_m], \\ D_+ &= (n-1)[(U(b,0) - U(0,\bar{e}))((2n-3)f_m^2 + \max\{f'_{\max}, 0\}) - 2\underline{\mu}(\bar{e})f_m], \end{aligned} \tag{21}$$

Under Assumption A1, at least one of these bounds is finite. The following proposition provides the equilibrium existence result.

Proposition A1 *Suppose Assumption A1 is satisfied, U is regular, and the following conditions hold:*
(a) $-U_{ee}(v, e)$ is bounded away from zero, with $-U_{ee}(v, e) \geq c_0 > 0$ for all $(v, e) \in [0, b] \times [0, \bar{e}]$.

(b) $c_0 > D \equiv \min\{D_-, D_+\}$*, where* D_- *and* D_+ *are given by (21).*

(c) $(n-1)f_m < \min\left\{\frac{c_0}{-\underline{\mu}(\bar{e})}, \frac{\bar{\mu}(\bar{e})}{U(0,\bar{e})}, \frac{-\bar{\mu}(\underline{e})}{U(b,0)-U(0,\bar{e})}\right\}$.

Then for any $\mathbf{v} \in \mathcal{V}_b$*,*

(i) $A(\mathbf{v}, e)$ *is strictly decreasing in* e *for* $e \in [0, \bar{e}]$*;*

(ii) there exists a unique $\hat{e} \in [0, \bar{e}]$ *solving (3), with* $\hat{e} = 0$ *if and only if* $\mathbf{v} = (\frac{b}{n}, \ldots, \frac{b}{n})$*;*

(iii) $\hat{e}$ *is the symmetric pure-strategy equilibrium in the tournament.*

The conditions in Proposition A1 generalize the respective conditions in Drugov and Ryvkin (2020) obtained for the risk-neutral separable case, $U(v, e) = v - c(e)$. Condition (a) restricts the generalized cost curvature, $U_{ee}(v, e)$, to be bounded away from zero. Condition (b) comes from the fact that $D - c_0$ is an upper bound on the curvature of expected utility $W(e, \hat{e})$ in (1). Hence, conditions (a) and (b) ensure that the player's problem is concave in e. Bound D used in condition (b) is defined as the minimum of D_- and D_+ because both of them can serve as upper bounds, and hence, one only of them has to be finite. Indeed, the second derivative of $\pi^{(r)}(e, \hat{e})$ in (2) can be represented in two ways, see Eqs. (26) and (27) in the proof.

Finally, condition (c) consists of three restrictions corresponding to the three terms subjected to the min operator. The first restriction is sufficient for part (i). The second one ensures that $A(\mathbf{v}, \bar{e}) < 0$, which, together with part (i) and $A(\mathbf{v}, 0)$ being always positive leads to part (ii). The third one comes from the players' participation constraint in the equilibrium. Note that both conditions (b) and (c) become tighter as the number of players increases or noise dispersion decreases.

Comparing Proposition A1 to Propositon 10 in Drugov and Ryvkin (2020) one notices that parts (i) and (ii)—and hence, terms $\frac{c_0}{-\underline{\mu}(\bar{e})}$ and $\frac{\bar{\mu}(\bar{e})}{U(0,\bar{e})}$ in condition (c)—are new. Indeed, in the risk neutral separable case, from (16), $A(\mathbf{v}, e) = \sum_{r=1}^{n} \beta_r v_r - c'(e)$ strictly decreases in e and crosses zero without any further conditions. It is also clear that in terms of equilibrium existence the risk averse separable case is similar to the risk neutral case, and it is the nonseparability that makes the analysis more involved.

Throughout this paper, we only consider monotone prize schedules. Apart from obvious managerial reasons for this restriction, there are technical reasons for the exclusion of nonmonotone prize schedule due to the equilibrium existence considerations. While in principle a symmetric equilibrium may exist in some cases when $\mathbf{v}$ is nonmonotone, it cannot exist for all nonmonotone prize schedules. This can be seen already in the risk neutral separable case by writing $A(\mathbf{v}, e)$ in the form $A(\mathbf{v}, e) = \sum_{r=1}^{n-1} B_r(v_r - v_{r+1}) - c'(e)$, where $B_r > 0$ for all $r = 1, \ldots, n-1$ (see (22) below for the definition of B_r). It is then easy to construct prize schedules for which $A(\mathbf{v}, e) < 0$ for all e.

Proof of Proposition A1

Step (i): Sufficient conditions for $A(\mathbf{v}, e) = 0$ to have a unique solution $\hat{e} \in [0, \bar{e}]$.

For a unique solution to exists, it is sufficient to ensure that $A(\mathbf{v}, 0) > 0$, $A(\mathbf{v}, \bar{e}) < 0$ and $A_e(\mathbf{v}, e) < 0$ for all $e \in (0, \bar{e})$. Let $B_r = \sum_{k=1}^{r} \beta_k$ denote the cumulative version of coefficients β_r. It can be directly checked that

$$B_r = r\binom{n-1}{r} \int F(x)^{n-r-1}[1-F(x)]^{r-1} f(x) dF(x). \tag{22}$$

which implies $B_r > 0$ for $r = 1, \ldots, n-1$ and $B_n = 0$. Using summation by parts, we can write

$$A(\mathbf{v}, e) = \sum_{r=1}^{n-1} B_r[U(v_r, e) - U(v_{r+1}, e)] + \frac{1}{n} \sum_{r=1}^{n} U_e(v_r, e),$$

which immediately gives $A(\mathbf{v}, 0) \geq 0$, with equality if and only if v_r is independent of r (i.e., $\mathbf{v}$ is the constant prize schedule with $v_r = \frac{b}{n}$). Furthermore,

$$\begin{aligned}
A(\mathbf{v}, \bar{e}) &= \sum_{r=1}^{n-1} B_r[U(v_r, \bar{e}) - U(v_{r+1}, \bar{e})] + \frac{1}{n} \sum_{r=1}^{n} U_e(v_r, \bar{e}) \\
&\leq \sum_{r=1}^{n-1} B_r[U(b, \bar{e}) - U(0, \bar{e})] + \max_{v \in [0,b]} U_e(v, \bar{e}) = -U(0, \bar{e}) \sum_{r=1}^{n-1} B_r + \max_{v \in [0,b]} U_e(v, \bar{e}).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{r=1}^{n-1} B_r &= \int \sum_{r=1}^{n-1} \frac{(n-1)!}{(n-1-r)!(r-1)!} F(x)^{n-1-r}[1-F(x)]^{r-1} f(x) dF(x) \\
&= \int \sum_{r=0}^{n-2} \frac{(n-1)!}{(n-2-r)!r!} F(x)^{n-2-r}[1-F(x)]^{r} f(x) dF(x) \\
&= (n-1) \int f(x) dF(x) \leq (n-1) f_m.
\end{aligned}$$

Thus, for $A(\mathbf{v}, \bar{e}) < 0$ it is sufficient to require that $(n-1) f_m < \frac{\max_{v \in [0,b]} U_e(v, \bar{e})}{U(0, \bar{e})} = \frac{\bar{\mu}(e)}{U(0, \bar{e})}$.

Next, we analyze $A_e(\mathbf{v}, e)$:

$$\begin{aligned}A_e(\mathbf{v}, e) &= \sum_{r=1}^{n-1} B_r[U_e(v_r, e) - U_e(v_{r+1}, e)] + \frac{1}{n}\sum_{r=1}^{n} U_{ee}(v_r, e)\\ &\leq \sum_{r=1}^{n-1} B_r[U_e(v_r, 0) - U_e(v_{r+1}, \bar{e})] - c_0 \leq -\min_{v\in[0,b]} U_e(v, \bar{e})(n-1)f_m - c_0.\end{aligned}$$

Thus, for $A_e(\mathbf{v}, e) < 0$ it is sufficient to require that $(n-1)f_m < \frac{c_0}{-\min_{v\in[0,b]} U_e(v,\bar{e})} = \frac{c_0}{-\underline{\mu}(\bar{e})}$.

Step (ii): Sufficient conditions for $\hat{e}$ to be an equilibrium

We will identify conditions for $W(e, \hat{e})$ to be strictly concave in e for all $e \in [0, \bar{e}]$. From (1),

$$W_{ee} = \sum_{r=1}^{n} \left[\pi_{ee}^{(r)}(e, \hat{e})U(v_r, e) + 2\pi_e^{(r)}(e, \hat{e})U_e(v_r, e) + \pi^{(r)}(e, \hat{e})U_{ee}(v_r, e)\right]. \tag{23}$$

Let $\Pi^{(r)}(e, \hat{e}) = \sum_{k=1}^{r} \pi^{(k)}(e, \hat{e})$ denote the cumulative version of probabilities $\pi^{(r)}(e, \hat{e})$. By construction, $\Pi^{(n)}(e, \hat{e}) = 1$, and hence $\Pi_e^{(n)}(e, \hat{e}) = \Pi_{ee}^{(n)}(e, \hat{e}) = 0$ for all e. Using summation by parts and the uniform bound $U_{ee} \leq -c_0$, (23) can be rewritten as

$$W_{ee} \leq \sum_{r=1}^{n-1} \left[\Pi_{ee}^{(r)}(e, \hat{e})(U(v_r, e) - U(v_{r+1}, e)) + 2\Pi_e^{(r)}(e, \hat{e})(U_e(v_r, e) - U_e(v_{r+1}, e))\right] - c_0. \tag{24}$$

Let $\Delta e = e - \hat{e}$. From (2),

$$\begin{aligned}\pi_e^{(r)}(e, \hat{e}) = \binom{n-1}{r-1} \int F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-2}\\ \times[(n-r)(1 - F(\Delta e + x)) - (r-1)F(\Delta e + x)]f(\Delta e + x)dF(x),\end{aligned} \tag{25}$$

and it can be directly verified that, similar to (22),

$$\Pi_e^{(r)}(e, \hat{e}) = r\binom{n-1}{r} \int F(\Delta e + x)^{n-r-1}[1 - F(\Delta e + x)]^{r-1} f(\Delta e + x)dF(x). \tag{26}$$

Changing the variable of integration as $\Delta e + x \to x$, (26) can be written as

$$\Pi_e^{(r)}(e,\hat{e}) = r\binom{n-1}{r}\int F(x)^{n-r-1}[1-F(x)]^{r-1}f(x-\Delta e)dF(x) \tag{27}$$
$$\leq f_m r\binom{n-1}{r}\int F(x)^{n-1-r}[1-F(x)]^{r-1}dF(x).$$

The inequality follows because $f(x-\Delta e)$ does not exceed f_m. Next, we differentiate (27) once again to calculate $\Pi_{ee}^{(r)}(e,\hat{e})$. Note that $f(x-\Delta e)$ is not necessarily continuous in e for all $x \in \mathcal{X}$. Indeed, if $\underline{x}$ is finite and $f(\underline{x}) > 0$ then $f(x-\Delta e)$ has a jump discontinuity at $\Delta e = x - \underline{x}$, and similarly for $\overline{x}$. Therefore, to preserve the continuity (and hence, differentiability) of the integrand in (27), the interval of integration must be changed to $[\underline{x}+\Delta e, \overline{x}]$ for $\Delta e > 0$, to $[\underline{x}, \overline{x}+\Delta e]$ for $\Delta e < 0$, and the case of $\Delta e = 0$ needs to be considered separately. This gives

$$\Pi_{ee}^{(r)}(e,\hat{e}) = r\binom{n-1}{r}\Big[-\int F(x)^{n-r-1}[1-F(x)]^{r-1}f'(x-\Delta e)dF(x)$$
$$- F(\underline{x}+\Delta e)^{n-r-1}[1-F(\underline{x}+\Delta e)]^{r-1}f(\underline{x})f(\underline{x}+\Delta e)\mathbb{1}_{\Delta e>0}$$
$$+ F(\overline{x}+\Delta e)^{n-r-1}[1-F(\overline{x}+\Delta e)]^{r-1}f(\overline{x})f(\overline{x}+\Delta e)\mathbb{1}_{\Delta e<0}$$
$$- f(\underline{x})^2\mathbb{1}_{r=n-1}\mathbb{1}_{\Delta e=0} + f(\overline{x})^2\mathbb{1}_{r=1}\mathbb{1}_{\Delta e=0}\Big]$$
$$\leq r\binom{n-1}{r}\Big[\max\{-f'_{\min},0\}\int F(x)^{n-r-1}[1-F(x)]^{r-1}dF(x)$$
$$+ f_m^2 F(\overline{x}+\Delta e)^{n-r-1}[1-F(\overline{x}+\Delta e)]^{r-1}\mathbb{1}_{\Delta e<0}\Big] + (n-1)f_m^2\mathbb{1}_{\Delta e=0}. \tag{28}$$

Here, $\mathbb{1}_S$ is the indicator equal to one if S is true and zero otherwise. The inequality in (28) follows from the following considerations: (i) $-f'(x-\Delta e)$ does not exceed $-f'_{\min}$; (ii) all negative terms can be ignored; (iii) in the remaining two positive terms the product of two pdfs does not exceed f_m^2.

In order to transform (24), we need to sum up the right-hand side of (27) over $r = 1,\ldots,n-1$. Note that for any $z \in [0,1]$ we have

$$\sum_{r=1}^{n-1} r\binom{n-1}{r}z^{n-r-1}(1-z)^{r-1} = \sum_{r=1}^{n-1}\frac{(n-1)!}{(n-1-r)!(r-1)!}z^{n-r-1}(1-z)^{r-1}$$
$$= \sum_{r=0}^{n-2}\frac{(n-1)!}{(n-2-r)!r!}z^{n-2-r}(1-z)^r = n-1.$$

The inequality (24) then gives

$$
\begin{aligned}
&W_{ee} \leq [U(b,0) - U(0,\bar{e})][\max\{-f'_{\min}, 0\}(n-1) + f_m^2(n-1)\mathbb{1}_{\Delta e<0} + f_m^2(n-1)^2\mathbb{1}_{\Delta e=0}] \\
&- 2\min_{v\in[0,b]} U_e(v,\bar{e})f_m(n-1) - c_0 \\
&\leq (n-1)[(U(b,0) - U(0,\bar{e}))((n-1)f_m^2 + \max\{-f'_{\min}, 0\}) - 2\min_{v\in[0,b]} U_e(v,\bar{e})f_m] - c_0,
\end{aligned}
$$

where the second inequality follows because $\Delta e < 0$ and $\Delta e = 0$ cannot hold at the same time. Thus, in order for $W_{ee} < 0$ to hold it is sufficient that $c_0 > D_-$, with D_- given by (21).

An alternative upper bound on $\Pi_{ee}^{(r)}(e,\hat{e})$ can be obtained by differentiating (26) directly. Taking into account the variable limits of integration, to preserve the differentiability of the integrand similar to (28), this gives

$$
\begin{aligned}
&\Pi_{ee}^{(r)}(e,\hat{e}) = r\binom{n-1}{r}\Bigg[\int F(\Delta e+x)^{n-r-2}[1-F(\Delta e+x)]^{r-2} \\
&\times [(n-r-1)(1-F(\Delta e+x)) - (r-1)F(\Delta e+x)]f(\Delta e+x)^2 dF(x) \\
&+ \int F(\Delta e+x)^{n-r-1}[1-F(\Delta e+x)]^{r-1}f'(\Delta e+x)dF(x) \\
&- F(\Delta e+\underline{x})^{n-r-1}[1-F(\Delta e+\underline{x})]^{r-1}f(\Delta e+\underline{x})f(\underline{x})\mathbb{1}_{\Delta e>0} \\
&+ F(\Delta e+\overline{x})^{n-r-1}[1-F(\Delta e+\overline{x})]^{r-1}f(\Delta e+\overline{x})f(\overline{x})\mathbb{1}_{\Delta e<0} \\
&- f(\underline{x})^2\mathbb{1}_{r=n-1}\mathbb{1}_{\Delta e=0} + f(\overline{x})^2\mathbb{1}_{r=1}\mathbb{1}_{\Delta e=0}\Bigg] \\
&\leq r\binom{n-1}{r}\Bigg[(n-r-1)f_m^2\int F(\Delta e+x)^{n-r-2}[1-F(\Delta e+x)]^{r-1}dF(x) \\
&+ \max\{f'_{\max}, 0\}\int F(\Delta e+x)^{n-r-1}[1-F(\Delta e+x)]^{r-1}dF(x) \\
&+ f_m^2F(\Delta e+x)^{n-r-1}[1-F(\Delta e+x)]^{r-1}\mathbb{1}_{\Delta e<0}\Bigg] + (n-1)f_m^2\mathbb{1}_{\Delta e=0}
\end{aligned}
\tag{29}
$$

As above, we omitted negative terms and replaced f and f' with the corresponding upper bounds. In order to sum up the right-hand side of (29), note that for any $z \in [0,1]$, we have

$$\sum_{r=1}^{n-1} r(n-r-1)\binom{n-1}{r} z^{n-2-r}(1-z)^{r-1} = \sum_{r=1}^{n-2} \frac{(n-1)!}{(n-r-2)!(r-1)!} z^{n-2-r}(1-z)^{r-1}$$
$$= \sum_{r=0}^{n-3} \frac{(n-1)!}{(n-3-r)!r!} z^{n-3-r}(1-z)^{r} = (n-1)(n-2).$$

Therefore, from (24),

$$W_{ee} \leq [U(b,0)-U(0,\bar{e})][(n-1)(n-2)f_m^2 + (n-1)\max\{f'_{\max},0\} + (n-1)f_m^2 \mathbb{1}_{\Delta e<0}$$
$$+ (n-1)^2 f_m^2 \mathbb{1}_{\Delta e=0}] - 2\min_{v\in[0,b]} U_e(v,\bar{e})(n-1)f_m - c_0$$
$$\leq (n-1)[(U(b,0)-U(0,\bar{e}))((2n-3)f_m^2 + \max\{f'_{\max},0\}) - 2\min_{v\in[0,b]} U_e(v,\bar{e})f_m] - c_0.$$

Thus, it is sufficient to require that $c_0 < D_+$, with D_+ given by (21). Both bounds are sufficient, and hence condition $c_0 > \min\{D_-, D_+\}$ guarantees that $W(e,\hat{e})$ is strictly concave in e for $e \in [0,\bar{e}]$.

Step (iii): Sufficient conditions for the participation constraint $W(\hat{e},\hat{e}) \geq 0$

From symmetry, $W(\hat{e},\hat{e}) = \frac{1}{n}\sum_{r=1}^{n} U(v_r,\hat{e})$. Recall that $\hat{e}$ satisfies the equation $A(\mathbf{v},e) = 0$, i.e., $\sum_{r=1}^{n-1} B_r[U(v_r,\hat{e})-U(v_{r+1},\hat{e})] = \psi(\mathbf{v},\hat{e})$, where $\psi(\mathbf{v},e) = -\frac{1}{n}\sum_{r=1}^{n} U_e(v_r,e)$. Function ψ is strictly increasing in e and continuous in $\mathbf{v}$. Let $\psi^{-1}(\mathbf{v},t)$ denote the inverse of ψ with respect to e, which is continuous in $\mathbf{v}$ and strictly increasing in t. We have $\sum_{r=1}^{n-1} B_r \leq (n-1)f_m$; therefore,

$$\sum_{r=1}^{n-1} B_r[U(v_r,\hat{e}) - U(v_{r+1},\hat{e})] \leq [U(b,0)-U(0,\bar{e})](n-1)f_m.$$

This gives

$$\hat{e} = \psi^{-1}(\mathbf{v}, \sum_{r=1}^{n-1} B_r[U(v_r,\hat{e}) - U(v_{r+1},\hat{e})]) \leq \psi^{-1}(\mathbf{v}, [U(b,0)-U(0,\bar{e})](n-1)f_m) \equiv e_0.$$

Effort e_0 is an upper bound on the equilibrium effort. It is, therefore, sufficient to impose a condition such that $\sum_{r=1}^{n} U(v_r,e_0) \geq 0$.

From the definition of $\underline{e}$ we know that for any feasible prize schedule $\frac{1}{n}\sum_{r=1}^{n} U(v_r,e) \geq 0$ for any $e \leq \underline{e}$. Therefore, it is sufficient to require that $e_0 \leq \underline{e}$ or, equivalently, $[U(b,0)-$

$U(0,\bar{e})](n-1)f_m \le \psi(\mathbf{v},\underline{e})$, and hence $[U(b,0)-U(0,\bar{e})](n-1)f_m \le -\max_{v\in[0,b]} U_e(v,\underline{e})$ is sufficient. A combination of this condition with the conditions for $A(\mathbf{v},\bar{e}) < 0$ and $A_e(\mathbf{v},e) < 0$ from step (i) produces condition (c) of the proposition. ■

Appendix B: Proofs

Proof of Lemma 1

Part (i) follows directly from Eq. (4).

For the rest of the proof, let $F^{-1}(z) = \inf\{x \in \mathcal{X} : F(x) \ge z\}$ denote the quantile function of noise, and let $m(z) = f(F^{-1}(z)) : [0,1] \to \mathbb{R}_+$ denote the inverse quantile density (Parzen, 1979), which is continuous, piece-wise differentiable and integrable due to the properties of $f(\cdot)$. After the probability integral change of variable, $z = F(x)$, Eq. (4) becomes

$$\beta_r = \binom{n-1}{r-1} \int_0^1 z^{n-r-1}(1-z)^{r-2}[n-r-(n-1)z]m(z)dz. \tag{30}$$

Integrating Eq. (30) by parts, obtain

$$\begin{aligned}
\beta_r &= \binom{n-1}{r-1} \int_0^1 m(z) z^{n-r-1}(1-z)^{r-2}[n-r-(n-1)z]dz \\
&= \binom{n-1}{r-1} \int_0^1 m(z) d[z^{n-r}(1-z)^{r-1}] \\
&= \binom{n-1}{r-1} \left[m(z) z^{n-r}(1-z)^{r-1}\Big|_0^1 - \int_0^1 z^{n-r}(1-z)^{r-1} m'(z)dz \right] \\
&= \binom{n-1}{r-1} [m(1)\mathbb{1}_{r=1} - m(0)\mathbb{1}_{r=n}] - \frac{1}{n}\frac{n!}{(n-r)!(r-1)!} \int_0^1 z^{n-r}(1-z)^{r-1} m'(z)dz \\
&= m(1)\mathbb{1}_{r=1} - m(0)\mathbb{1}_{r=n} - \frac{1}{n}\mathrm{E}(m'(Z_{n+1-r:n})).
\end{aligned} \tag{31}$$

Here, $Z_{n+1-r:n}$ are order statistics of the uniform distribution on $[0,1]$. These order statistics are FOSD-decreasing in r.

Part (ii): note that if $f(\cdot)$ is (strictly) log-concave then $m(\cdot)$ is (strictly) concave, and hence $m'(z)$ is (strictly) decreasing and the expectation in (31) is (strictly) increasing in r. The first two terms in (31) give $m(1)$ for $r=1$, $-m(0)$ for $r=n$ and 0 otherwise; hence, combined we have a sequence that is (strictly) decreasing in r.

Part (iii): Note that if $f(\cdot)$ is (strictly) log-convex then $m(\cdot)$ is (strictly) convex, and

$f(\overline{x}) = 0$ implies $m(1) = 0$. Eq. (31) then gives a sequence that is (strictly) increasing in r for $r \leq n-1$.

We first prove part (v), and then go back to part (iv).

Part (v): Note that if $f(\cdot)$ is unimodal, then $m(\cdot)$ is also unimodal and hence $m'(\cdot)$ is single-crossing $+-$; that is, there exists a $\hat{z} \in [0,1]$ such that $m'(z) \geq 0$ for $z \leq \hat{z}$ and $m'(z) \leq 0$ for $z \geq \hat{z}$. The cases of monotone $m'(\cdot)$ are covered in parts (ii) and (iii). Suppose $m'(\cdot)$ is nonmonotone.

We know from part (i) that $\beta_1 > 0$ and $\beta_n < 0$; therefore, for $n \leq 3$ the result is trivial (and does not require unimodality). Suppose $n \geq 4$, and consider some r such that $3 \leq r \leq n-1$ and $\beta_r > 0$. It is sufficient to show that $\beta_{r-1} > 0$. From (31),

$$\begin{aligned}
&\beta_{r-1} = -\binom{n-1}{r-2}\int_0^1 z^{n+1-r}(1-z)^{r-2}m'(z)dz \\
&= -\frac{r-1}{n+1-r}\binom{n-1}{r-1}\int_0^1 \frac{z}{1-z}z^{n-r}(1-z)^{r-1}m'(z)dz \\
&= -\frac{r-1}{n+1-r}\binom{n-1}{r-1}\left[\int_0^{\hat{z}} \frac{z}{1-z}z^{n-r}(1-z)^{r-1}m'(z)dz + \int_{\hat{z}}^1 \frac{z}{1-z}z^{n-r}(1-z)^{r-1}m'(z)dz\right] \\
&\geq -\frac{r-1}{n+1-r}\binom{n-1}{r-1}\left[\frac{\hat{z}}{1-\hat{z}}\int_0^{\hat{z}} z^{n-r}(1-z)^{r-1}m'(z)dz + \frac{\hat{z}}{1-\hat{z}}\int_{\hat{z}}^1 z^{n-r}(1-z)^{r-1}m'(z)dz\right] \\
&= -\frac{r-1}{n+1-r}\binom{n-1}{r-1}\frac{\hat{z}}{1-\hat{z}}\int_0^1 z^{n-r}(1-z)^{r-1}m'(z)dz = \frac{r-1}{n+1-r}\frac{\hat{z}}{1-\hat{z}}\beta_r > 0.
\end{aligned}$$

The inequality on the fourth line follows because $\frac{z}{1-z}$ is positive and increasing and $m'(z)$ is positive (negative) in the first (second) integral.

Part (iv): It follows that $m(\cdot)$ is first concave, then convex, which implies $m'(\cdot)$ is U-shaped and hence $m''(\cdot)$ is single-crossing $-+$. In order to show that β_r is unimodal, we will show that $\beta_r - \beta_{r-1}$ is single-crossing $+-$. Assume $n-1 \geq r \geq 2$ and recall that

$m(1) = 0$; using (31),

$$\begin{aligned}
&\beta_r - \beta_{r-1} = -\frac{(n-1)!}{(n-r)!(r-1)!}\int_0^1 z^{n-r}(1-z)^{r-1}m'(z)dz \\
&+ \frac{(n-1)!}{(n-r+1)!(r-2)!}\int_0^1 z^{n-r+1}(1-z)^{r-2}m'(z)dz \\
&= -\frac{(n-1)!}{(n-r+1)!(r-1)!}\int_0^1 z^{n-r}(1-z)^{r-2}[(n-r+1)(1-z)-(r-1)z]m'(z)dz \\
&= -\frac{(n-1)!}{(n-r+1)!(r-1)!}\int_0^1 m'(z)d[z^{n-r+1}(1-z)^{r-1}] \\
&= \frac{(n-1)!}{(n-r+1)!(r-1)!}\int_0^1 z^{n-r+1}(1-z)^{r-1}m''(z)dz, \quad r \leq n-1.
\end{aligned}$$

where the last line follows from integration by parts. For $r = n$, we have

$$\beta_n - \beta_{n-1} = -m(0) + \int_0^1 z(1-z)^{n-1}m''(z)dz.$$

The result then follows from the steps similar to the ones in the proof of part (v). ∎

Proof of Proposition 1

Part (i): suppose not, i.e., there exists a $\tilde{\mathbf{v}} \in \mathcal{V}$ such that $A(\tilde{\mathbf{v}}, e^*) > A(\mathbf{v}^*, e^*) = 0$. But $A_e(\mathbf{v}, e) < 0$ under our assumptions, which implies there exists an $\tilde{e} > e^*$ such that $A(\tilde{\mathbf{v}}, \tilde{e}) = 0$. This contradicts to e^* being optimal in problem (5).

Part (ii): Let us first prove that there exists an e^* such that $A(\mathbf{v}^*, e^*) = 0$ for all $\mathbf{v}^* \in \mathbf{v}^*(e^*)$. Consider an arbitrary $e^1 \in [0, \bar{e}]$, pick a $\mathbf{v}^1 \in \mathbf{v}^*(e^1)$, and suppose $A(\mathbf{v}^1, e^1) > 0$ (the opposite case will be considered below). It follows from $A_e < 0$ that there exists an $e^2 > e^1$ such that $A(\mathbf{v}^1, e^2) = 0$. Picking a $\mathbf{v}^2 \in \mathbf{v}^*(e^2)$, we have $A(\mathbf{v}^2, e^2) \geq A(\mathbf{v}^1, e^2) = 0$ because $\mathbf{v}^2$ maximizes $A(\mathbf{v}, e^2)$ over $\mathbf{v} \in \mathcal{V}$. If $A(\mathbf{v}^2, e^2) > 0$, then there exists an $e^3 > e^2$ such that $A(\mathbf{v}^2, e^3) = 0$. Continuing similarly, we obtain a sequence $\{e^k\}$ that is strictly increasing and bounded (by $\bar{e}$), and hence it converges to a limit e^* such that $A(\mathbf{v}^*, e^*) \geq 0$ for all $\mathbf{v}^* \in \mathbf{v}^*(e^*)$. Moreover, $A(\mathbf{v}^*, e^*) = 0$ because, otherwise, the process can be repeated starting with $e^1 = e^*$.

Suppose now that $A(\mathbf{v}^1, e^1) < 0$. Then, due to $A_e < 0$, there exists an $e^2 < e^1$ such that $A(\mathbf{v}^1, e^2) = 0$. Picking a $\mathbf{v}^2 \in \mathbf{v}^*(e^2)$, we have $A(\mathbf{v}^2, e^2) \geq 0$, and a converging sequence can be constructed as above.

We now show that, for $\mathbf{v}^* \in \mathbf{v}^*(e^*)$, $(\mathbf{v}^*, e^*)$ solves (5). Suppose not, i.e., there exists a $\tilde{\mathbf{v}} \in \mathcal{V}$ and $\tilde{e}$ such that $A(\tilde{\mathbf{v}}, \tilde{e}) = 0$ and $\tilde{e} > e^*$. Then, from $A_e < 0$ it follows that

$A(\tilde{\mathbf{v}}, e^*) > 0 = A(\mathbf{v}^*, e^*)$, which contradicts to $\mathbf{v}^*$ maximizing $A(\mathbf{v}, e^*)$.

Finally, we show that e^* satisfying $A(\mathbf{v}^*, e^*) = 0$ for all $\mathbf{v}^* \in \mathbf{v}^*(e^*)$ is unique. Suppose not, i.e., there are e^* and $\tilde{e}$ such that $e^* > \tilde{e}$ and $A(\mathbf{v}^*, e^*) = A(\tilde{\mathbf{v}}, \tilde{e}) = 0$ for all $\mathbf{v}^* \in \mathbf{v}^*(e^*)$ and all $\tilde{\mathbf{v}} \in \mathbf{v}^*(\tilde{e})$. By construction, $0 = A(\tilde{\mathbf{v}}, \tilde{e}) \geq A(\mathbf{v}^*, \tilde{e})$; however, from $A_e < 0$ it follows that $0 = A(\mathbf{v}^*, e^*) < A(\mathbf{v}^*, \tilde{e})$—a contradiction. ■

Proof of Proposition 2

If $\mathbf{v}^*$ solves (7), it also maximizes Lagrangian $\mathcal{L}^{(0)}(\mathbf{v}, \lambda^*; e^*)$. Thus, $v_r \in q(\beta_r, \lambda^*; e^*)$. The objective in (10) satisfies strictly increasing differences in (β, v). It then follows from the monotone comparative statics that $\beta_r > \beta_{r'}$ implies then $v_r^* \geq v_{r'}^*$; moreover, $v_r^* > v_{r'}^*$ if $v_r^* > 0$.

When β_r is weakly decreasing, there may be solutions $\mathbf{v}^*$ that are not decreasing, but there has to be a weakly decreasing selection. Indeed, if $\beta_r = \beta_{r+1}$ for some r, we have $q(\beta_r, \lambda^*, e^*) = q(\beta_{r+1}, \lambda^*, e^*)$; therefore, if there is a solution $\mathbf{v}^*$ such that $v_r^* < v_{r+1}^*$, there is also a solution with $v_r^* > v_{r+1}^*$ where the two prizes are swapped. ■

Proof of Proposition 3

It is convenient to define, superficially, $v_{n+1} = 0$, and introduce nonnegative prize differentials, $d_r = v_r - v_{r+1}$. We have $v_r = \sum_{k=r}^n d_k$ for $r = 1, \ldots, n$, and, using summation by parts, the budget constraint takes the form $\sum_{r=1}^n r d_r = b$. In the new variables, the optimization problem (6), with $e = e^*$, becomes

$$\max A(\mathbf{v}, e^*), \quad \text{s.t. } \sum_{r=1}^n r d_r = b, \quad d_1, \ldots, d_n \geq 0. \tag{32}$$

The Lagrangian is

$$\mathcal{L}(\mathbf{v}, \lambda; e^*) = A(\mathbf{v}, e^*) + \lambda\left(b - \sum_{r=1}^n r d_r\right),$$

producing the KT conditions

$$\sum_{k=1}^r \left[\beta_k U_v(v_k, e^*) + \frac{1}{n} U_{ev}(v_k, e^*)\right] \leq r\lambda, \quad \text{with equality if } v_r > v_{r+1}, \quad r = 1, \ldots, n.$$

As before, due to the linearity of the constraints, the KT necessity theorem implies that if $\mathbf{v}$ solves (32) then there exists a $\lambda^* > 0$ such that $\mathbf{v}$ satisfies the KT conditions.

Optimal prizes have a step-wise decreasing structure, with critical points $1 \leq r_1 < \ldots < r_K \leq n$ such that v_r is strictly decreasing in r at the critical points (as long as prizes are positive) and remains constant between them. Therefore, the KT conditions hold with equality at the critical points, and (generically) as inequalities in between, as long as prizes remain positive. Since $v_1 > 0$ always holds, the first r_1 KT conditions have the form

$$
\begin{aligned}
&U_v(v_1, e^*) \sum_{l=1}^{k} \beta_l + \frac{k}{n} U_{ev}(v_1, e^*) \le k\lambda, \quad k = 1, \dots, r_1 - 1, \\
&U_v(v_1, e^*) \sum_{l=1}^{r_1} \beta_l + \frac{r_1}{n} U_{ev}(v_1, e^*) = r_1\lambda.
\end{aligned}
$$

Using coefficients $\bar{\beta}_{r:r'} = \frac{1}{r'-r+1} \sum_{l=r}^{r'} \beta_l$, these can be written as

$$
\begin{aligned}
&\bar{\beta}_{1:k} U_v(v_1, e^*) + \frac{1}{n} U_{ev}(v_1, e^*) \le \lambda, \quad k = 1, \dots, r_1 - 1, \\
&\bar{\beta}_{1:r_1} U_v(v_1, e^*) + \frac{1}{n} U_{ev}(v_1, e^*) = \lambda.
\end{aligned}
\tag{33}
$$

This implies $\bar{\beta}_{1:k} \le \bar{\beta}_{1:r_1}$ for all $k = 1, \dots, r_1$, and hence we define $r_1 = \max\{r : \bar{\beta}_{1:k} \le \bar{\beta}_{1:r} \ \forall k = 1, \dots, r\}$.

Consider now the next group of KT conditions for $r = r+1, \dots, r_2$, assuming $v_{r_2} > 0$:

$$
\begin{aligned}
&U_v(v_1, e^*) \sum_{l=1}^{r_1} \beta_l + U_v(v_{r_2}, e^*) \sum_{l=r_1+1}^{k} \beta_l + \frac{r_1}{n} U_{ev}(v_1, e^*) + \frac{k - r_1}{n} U_{ev}(v_{r_2}, e^*) \le k\lambda, \\
&\qquad k = r_1 + 1, \dots, r_2 - 1, \\
&U_v(v_1, e^*) \sum_{l=1}^{r_1} \beta_l + U_v(v_{r_2}, e^*) \sum_{l=r_1+1}^{r_2} \beta_l + \frac{r_1}{n} U_{ev}(v_1, e^*) + \frac{r_2 - r_1}{n} U_{ev}(v_{r_2}, e^*) = r_2\lambda.
\end{aligned}
$$

Combining these with (33), obtain

$$
\begin{aligned}
&\bar{\beta}_{r_1+1:k} U_v(v_{r_2}, e^*) + \frac{1}{n} U_{ev}(v_{r_2}, e^*) \le \lambda, \quad k = r_1 + 1, \dots, r_2 - 1, \\
&\bar{\beta}_{r_1+1:r_2} U_v(v_{r_2}, e^*) + \frac{1}{n} U_{ev}(v_{r_2}, e^*) = \lambda.
\end{aligned}
\tag{34}
$$

We, therefore, define $r_2 = \max\{r : \bar{\beta}_{r_1+1:k} \le \bar{\beta}_{r_1+1:r} \ \forall k = r_1 + 1, \dots, r\}$. Continuing similarly, we obtain the sequence of critical points defined in (11).

Optimal prizes at the critical points, v_{r_k}, as long as they are positive, solve the equations

$$\bar{\beta}_{r_{k-1}+1:r_k} U_v(v_{r_k}, e^*) + \frac{1}{n} U_{ev}(v_{r_k}, e^*) = \lambda, \quad k = 1, \dots, s.$$

Here, s is the number of distinct positive prizes. By construction, coefficients $\bar{\beta}_{r_{k-1}+1:r_k}$ and prizes v_{r_k} are strictly decreasing in k for $k \leq s$. Moreover, setting $\lambda = \lambda^*$, the Lagrangian can be written in an additive separable form as

$$\mathcal{L}(\mathbf{v}, \lambda^*; e^*) = \sum_{k=1}^{K} (r_k - r_{k-1}) \left[\bar{\beta}_{r_{k-1}+1:r_k} U(v_{r_k}, e^*) + \frac{1}{n} U_e(v_{r_k}, e^*) - \lambda^* v_{r_k} \right] + \lambda^* b;$$

therefore, similar to Section 3.1, $v_{r_k} \in q(\bar{\beta}_{r_{k-1}+1:r_k}, \lambda^*; e^*)$. ■

Proof of Proposition 4

Suppose $v_n > 0$ and consider a modification of prizes $\mathbf{v} \to \mathbf{v}'$ such that $v'_1 = v_1 + v_n$ and $v'_n = 0$ (i.e., the entire prize v_n is transferred to v_1; the resulting prize schedule is feasible). This gives

$$\begin{aligned} A(\mathbf{v}', e) &= A(\mathbf{v}, e) + \beta_1 (U(v'_1, e) - U(v_1, e)) + \beta_n (U(0, e) - U(v_n, e)) \\ &+ \frac{1}{n} [U_e(v'_1, e) - U_e(v_1, e) + U_e(0, e) - U_e(v_n, e)] > A(\mathbf{v}, e). \end{aligned}$$

The terms with β_1 and β_n are strictly positive. The term multiplying $\frac{1}{n}$ is positive provided

$$U_e(v_n + v_1, e) - U_e(v_1, e) \geq U_e(v_n, e) - U_e(0, e),$$

which holds if U_e is convex in v or $U_{vve} \geq 0$.

Thus, $A(\mathbf{v}, e)$ can be strictly increased by shifting v_n to v_1; by the same argument, $A(\mathbf{v}, e)$ can be further increased by shifting v_{n-1} to v_1 if $v_{n-1} > 0$, and so on, finally reaching $v_{\hat{r}+1}$. The resulting $A(\mathbf{v}, e)$ is strictly greater than the initial one, and because it is strictly decreasing in e, the e^* solving $A(\mathbf{v}, e) = 0$ is larger. ■

Proof of Corollary 3

We need to show that $\beta_r \le 0$ for $r \ge 2$. From (4), coefficients β_r can be written as

$$
\begin{aligned}
\beta_r &= \frac{(n-1)!}{(n-1-r)!(r-1)!}\int_{\mathcal{X}} F(x)^{n-r-1}[1-F(x)]^{r-1}f(x)dF(x)\\
&- \frac{(n-1)!}{(n-r)!(r-2)!}\int_{\mathcal{X}} F(x)^{n-r}[1-F(x)]^{r-2}f(x)dF(x)\\
&= \mathrm{E}(f(X_{n-r:n-1})) - \mathrm{E}(f(X_{n-r+1:n-1})),
\end{aligned}
$$

Order statistics $X_{n-r:n-1}$ are FOSD-decreasing in r, and the result follows immediately because $f(\cdot)$ is an increasing function. ■

Lemma B1 *Define $\bar{e}$ by the condition $U(1,\bar{e}) = 0$. Suppose $e^* \in [0,\bar{e}]$ is optimal under U with $b = 1$, $\tilde{U}$ is regular, and there exists b large enough such that*

$$\beta_1[\tilde{U}(b,\bar{e}) - \tilde{U}(0,\bar{e})] \ge -\tilde{U}_e(0,\bar{e}) - \frac{1}{n}[\tilde{U}_e(b,\bar{e}) - \tilde{U}_e(0,\bar{e})]. \tag{35}$$

Then there exists a budget $b^c > 0$ such that e^* is optimal under $\tilde{U}$ with budget $b = b^c$.[20]

Proof Consider problem (6) with utility $\tilde{U}$ and $e = e^*$, and let $\tilde{A}^*(b,e^*)$ denote its optimal value function. Because $\mathcal{V}_b \subset \mathcal{V}_{b'}$ for $b < b'$, we have $\tilde{A}^*(b,e^*)$ increasing in b. It is sufficient to show that there exists a b^c such that $\tilde{A}^*(b^c,e^*) = 0$. From Berge's theorem, $\tilde{A}^*(b,e^*)$ is continuous in b. Moreover,

$$\tilde{A}^*(0,e^*) = \sum_{r=1}^{n}\beta_r\tilde{U}(0,e^*) + \frac{1}{n}\sum_{r=1}^{n}\tilde{U}_e(0,e^*) = \tilde{U}_e(0,e^*) < 0,$$

where we used the property $\sum_{r=1}^{n}\beta_r = 0$. Thus, it is sufficient to show that $\tilde{A}^*(b,e^*) \ge 0$ for a large enough b.

Consider the WTA prize schedule $\mathbf{v}_b^1 = (b,0,\dots,0)$. By construction, $\tilde{A}^*(b,e^*) \ge \tilde{A}(\mathbf{v}_b^1,e^*) \ge \tilde{A}(\mathbf{v}_b^1,\bar{e})$, and hence it is sufficient to require that $\tilde{A}(\mathbf{v}_b^1,\bar{e}) \ge 0$. We have

$$
\begin{aligned}
\tilde{A}(\mathbf{v}_b^1,\bar{e}) &= \beta_1\tilde{U}(b,\bar{e}) + \sum_{r=2}^{n}\beta_r\tilde{U}(0,\bar{e}) + \frac{1}{n}U_e(b,\bar{e}) + \frac{n-1}{n}U_e(0,\bar{e})\\
&= \beta_1[\tilde{U}(b,\bar{e}) - \tilde{U}(0,\bar{e})] + \frac{1}{n}[\tilde{U}_e(b,\bar{e}) - \tilde{U}_e(0,\bar{e})] + \tilde{U}_e(0,\bar{e}) \ge 0,
\end{aligned}
$$

[20] A weaker, albeit more complex, version of condition (35) is that effort $\bar{e}$ is implementable as optimal effort under $\tilde{U}$ with a large enough budget. Indeed, in this case $0 = \tilde{A}^*(b,\bar{e}) \le \tilde{A}^*(b,e^*)$ for some b, and Lemma B1 also holds.

from condition (35). ■

Proof of Proposition 5

Let e^* denote the (common) equilibrium effort. We start with the following lemma.

Lemma B2 *Suppose $\beta' > \beta$, $v' \in q(\beta', \lambda, e^*)$, $v \in q(\beta, \lambda, e^*)$, $\lambda > 0$, and $v > 0$. Then*

$$\frac{n\beta' + \gamma(v', e^*)}{n\beta + \gamma(v, e^*)} \frac{U_v(v', e^*)}{U_v(v, e^*)} \geq \frac{n\beta' + \tilde{\gamma}(v', e^*)}{n\beta + \tilde{\gamma}(v, e^*)} \frac{\tilde{U}_v(v', e^*)}{\tilde{U}_v(v, e^*)}. \tag{36}$$

Proof We know that $v' \geq v$. From the first-order condition (3) for v and v' we obtain,

$$\frac{n\beta' U_v(v', e^*) + U_{ve}(v', e^*)}{n\beta U_v(v, e^*) + U_{ve}(v, e^*)} = 1,$$

which can be rewritten as

$$\frac{n\beta' + \gamma(v', e^*)}{n\beta + \gamma(v, e^*)} \frac{U_v(v', e^*)}{U_v(v, e^*)} = 1. \tag{37}$$

Since $\tilde{U}$ is more risk averse than U, we have $\frac{U_v(v',e^*)}{U_v(v,e^*)} \geq \frac{\tilde{U}_v(v',e^*)}{\tilde{U}_v(v,e^*)}$.

Denote for brevity $\gamma \equiv \gamma(v, e^*)$ and $\gamma' \equiv \gamma(v', e^*)$. We know that $n\beta + \gamma = n\lambda > 0$, which implies, from the definition of $\tilde{U}$ being more risk averse, that $n\beta + \tilde{\gamma} > 0$ as well. Therefore, $\frac{n\beta'+\gamma'}{n\beta+\gamma} - \frac{n\beta'+\tilde{\gamma}'}{n\beta+\tilde{\gamma}}$ has the same sign as

$$(n\beta' + \gamma')(n\beta + \tilde{\gamma}) - (n\beta' + \tilde{\gamma}')(n\beta + \gamma) = (\tilde{\gamma} - \gamma)(n\beta' + \gamma') - (\tilde{\gamma}' - \gamma')(n\beta + \gamma) \geq 0.$$

The inequality follows since $n\beta' + \gamma' \geq n\beta + \gamma$ from (37), $\tilde{\gamma} \geq \gamma$ and $\tilde{\gamma}' \geq \gamma'$ from $\tilde{U}$ being more risk averse, and $\tilde{\gamma} - \gamma \geq \tilde{\gamma}' - \gamma'$ from condition (b) of the proposition. ■

The sequence of critical points $0 = r_0^*, r_1^*, \ldots, r_K^*$ is determined only by the distribution of noise; therefore, it is the same for both utility functions. Let $\lambda^*, \tilde{\lambda}^* > 0$ denote the corresponding optimal Lagrange multipliers for prize schedules $\mathbf{v}^*$ and $\tilde{\mathbf{v}}$.

Part (i): If K distinct positive prizes are optimal under $\tilde{U}$, we are done since at most K prizes can be optimal under U. Suppose $s < K$ distinct positive prizes are optimal under $\tilde{U}$, i.e., $\tilde{v}_{r_s^*} > \tilde{v}_{r_s^*+1} = 0$. It is sufficient to show that $v^*_{r_s^*+1} = 0$, i.e., it is impossible to have $s + 1$ or more distinct positive prizes under U. Suppose this is not true and

$v^*_{r^*_s+1} = v^*_{r^*_{s+1}} > 0$. Then $v^*_{r^*_s} > 0$ as well, and the corresponding KT conditions imply

$$\begin{aligned}
&\bar{\beta}_{r^*_{s-1}+1:r^*_s} U_v(v^*_{r^*_s}, e^*) + \frac{1}{n} U_{ve}(v^*_{r^*_s}, e^*) = \bar{\beta}_{r^*_s+1:r^*_{s+1}} U_v(v^*_{r^*_{s+1}}, e^*) + \frac{1}{n} U_{ve}(v^*_{r^*_{s+1}}, e^*) = \lambda^*, \\
&\bar{\beta}_{r^*_{s-1}+1:r^*_s} \tilde{U}_v(\tilde{v}_{r^*_s}, e^*) + \frac{1}{n} \tilde{U}_{ve}(\tilde{v}_{r^*_s}, e^*) = \tilde{\lambda}^* \geq \bar{\beta}_{r^*_s+1:r^*_{s+1}} \tilde{U}_v(0, e^*) + \frac{1}{n} \tilde{U}_{ve}(0, e^*).
\end{aligned}$$

The first equation implies that $n\bar{\beta}_{r^*_s+1:r^*_{s+1}} + \gamma(v^*_{r^*_{s+1}}, e^*) > 0$. The assumptions that $\tilde{\gamma} \geq \gamma$ and $\tilde{\gamma}_v \leq 0$ then imply that $n\bar{\beta}_{r^*_s+1:r^*_{s+1}} + \tilde{\gamma}(0, e^*) > 0$ as well. This gives

$$\frac{n\bar{\beta}_{r^*_{s-1}+1:r^*_s} + \gamma(v^*_{r^*_s}, e^*)}{n\bar{\beta}_{r^*_s+1:r^*_{s+1}} + \gamma(v^*_{r^*_{s+1}}, e^*)} \frac{U_v(v^*_{r^*_s}, e^*)}{U_v(v^*_{r^*_{s+1}}, e^*)} \leq \frac{n\bar{\beta}_{r^*_{s-1}+1:r^*_s} + \tilde{\gamma}(\tilde{v}_{r^*_s}, e^*)}{n\bar{\beta}_{r^*_s+1:r^*_{s+1}} + \tilde{\gamma}(0, e^*)} \frac{\tilde{U}_v(\tilde{v}_{r^*_s}, e^*)}{\tilde{U}_v(0, e^*)}. \tag{38}$$

Using Lemma B2, obtain

$$\frac{n\bar{\beta}_{r^*_{s-1}+1:r^*_s} + \tilde{\gamma}(v^*_{r^*_s}, e^*)}{n\bar{\beta}_{r^*_{s-1}+1:r^*_s} + \tilde{\gamma}(v^*_{r^*_{s+1}}, e^*)} \frac{\tilde{U}_v(v^*_{r^*_s}, e^*)}{\tilde{U}_v(v^*_{r^*_{s+1}}, e^*)} \leq \frac{n\bar{\beta}_{r^*_{s-1}+1:r^*_s} + \tilde{\gamma}(\tilde{v}_{r^*_s}, e^*)}{n\bar{\beta}_{r^*_s+1:r^*_{s+1}} + \tilde{\gamma}(0, e^*)} \frac{\tilde{U}_v(\tilde{v}_{r^*_s}, e^*)}{\tilde{U}_v(0, e^*)}.$$

Since $v^*_{r^*_{s+1}} > 0$ while $\tilde{U}_v$ and $\tilde{\gamma}$ are decreasing in v, it must be that $\tilde{v}_{r^*_s} \leq v^*_{r^*_s}$.

Applying the same arguments to prizes for ranks r^*_s and r^*_{s-1}, obtain $\tilde{v}_{r^*_{s-1}} \leq v^*_{r^*_{s-1}}$. Proceeding similarly, we obtain that $\tilde{v}_{r^*_k} \leq v^*_{r^*_k}$ for all $k = 1, \ldots, s$. This implies

$$\begin{aligned}
&(r^*_1 - r^*_0)\tilde{v}_{r^*_1} + \ldots + (r^*_s - r^*_{s-1})\tilde{v}_{r^*_s} \leq (r^*_1 - r^*_0)v^*_{r^*_1} + \ldots + (r^*_s - r^*_{s-1})v^*_{r^*_s} \\
&< (r^*_1 - r^*_0)v_{r^*_1} + \ldots + (r^*_s - r^*_{s-1})v_{r^*_s} + (r^*_{s+1} - r^*_s)v_{r^*_{s+1}} \leq 1,
\end{aligned}$$

giving $\sum_{r=1}^{\hat{r}} \tilde{v}_r < 1$, which is impossible.

Part (ii): Let s and $\tilde{s}$ denote the number of distinct positive prizes in $\mathbf{v}^*$ and $\tilde{\mathbf{v}}$, respectively (we showed in part (i) that $\tilde{s} \geq s$). We need to show that $\sum_{k=1}^r v^*_r \geq \sum_{k=1}^r \tilde{v}_r$ for all $r \leq r^*_{\tilde{s}}$. Suppose this inequality is not satisfied for some r, and let κ denote the lowest r such that it does not hold. Then it must be that $v^*_\kappa < \tilde{v}_\kappa$, or, equivalently, $v^*_{r^*_k} < \tilde{v}_{r^*_k}$ for some $k \leq \tilde{s}$. There are two possible cases: (a) $k < s$ and (b) $k \geq s$.

(a) Suppose $k < s$ and consider prizes at ranks r^*_k and r^*_{k+1}, which are both positive in $\mathbf{v}^*$ and in $\tilde{\mathbf{v}}$. From the KT conditions,

$$\begin{aligned}
&\bar{\beta}_{r^*_{k-1}+1:r^*_k} U_v(v^*_{r^*_k}, e^*) + \frac{1}{n} U_{ve}(v^*_{r^*_k}, e^*) = \bar{\beta}_{r^*_k+1:r^*_{k+1}} U_v(v^*_{r^*_{k+1}}, e^*) + \frac{1}{n} U_{ve}(v^*_{r^*_{k+1}}, e^*) = \lambda^*, \\
&\bar{\beta}_{r^*_{k-1}+1:r^*_k} \tilde{U}_v(\tilde{v}_{r^*_k}, e^*) + \frac{1}{n} \tilde{U}_{ve}(\tilde{v}_{r^*_k}, e^*) = \bar{\beta}_{r^*_k+1:r^*_{k+1}} \tilde{U}_v(\tilde{v}_{r^*_{k+1}}, e^*) + \frac{1}{n} \tilde{U}_{ve}(\tilde{v}_{r^*_{k+1}}, e^*) = \tilde{\lambda}^*.
\end{aligned}$$

This gives

$$\frac{n\bar{\beta}_{r^*_{k-1}+1:r^*_k}+\gamma(v^*_{r^*_k},e^*)}{n\bar{\beta}_{r^*_k+1:r^*_{k+1}}+\gamma(v^*_{r^*_{k+1}},e^*)}\frac{U_v(v^*_{r^*_k},e^*)}{U_v(v^*_{r^*_{k+1}},e^*)}=\frac{n\bar{\beta}_{r^*_{k-1}+1:r^*_k}+\tilde{\gamma}(\tilde{v}_{r^*_k},e^*)}{n\bar{\beta}_{r^*_k+1:r^*_{k+1}}+\tilde{\gamma}(\tilde{v}_{r^*_{k+1}},e^*)}\frac{\tilde{U}_v(\tilde{v}_{r^*_k},e^*)}{\tilde{U}_v(\tilde{v}_{r^*_{k+1}},e^*)}.$$

Using Lemma B2, obtain

$$\frac{n\bar{\beta}_{r^*_{k-1}+1:r^*_k}+\tilde{\gamma}(\tilde{v}_{r^*_k},e^*)}{n\bar{\beta}_{r^*_k+1:r^*_{k+1}}+\tilde{\gamma}(\tilde{v}_{r^*_{k+1}},e^*)}\frac{\tilde{U}_v(\tilde{v}_{r^*_k},e^*)}{\tilde{U}_v(\tilde{v}_{r^*_{k+1}},e^*)}\geq\frac{n\bar{\beta}_{r^*_{k-1}+1:r^*_k}+\tilde{\gamma}(v^*_{r^*_k},e^*)}{n\bar{\beta}_{r^*_k+1:r^*_{k+1}}+\tilde{\gamma}(v^*_{r^*_{k+1}},e^*)}\frac{\tilde{U}_v(v^*_{r^*_k},e^*)}{\tilde{U}_v(v^*_{r^*_{k+1}},e^*)}.$$

Since $v^*_{r^*_k}<\tilde{v}_{r^*_k}$, and $\tilde{U}_v$ and $\tilde{\gamma}$ are decreasing in v, it must be that $v^*_{r^*_{k+1}}\leq\tilde{v}_{r^*_{k+1}}$.

(b) Suppose $k\geq s$. If $\tilde{s}=s$, this is impossible (if the majorization inequality is violated for the last positive prize, then the budget constraint cannot hold); therefore, $\tilde{s}>s$. Then $v^*_{r^*_{k+1}}=0$ and $v^*_{r^*_{k+1}}\leq\tilde{v}_{r^*_{k+1}}$ holds automatically.

Thus, we obtained that a violation of the majorization inequality for some r implies that $v^*_{r^*_k}<\tilde{v}_{r^*_k}$ for some k, which in turn implies $v^*_{r^*_{k+1}}\leq\tilde{v}_{r^*_{k+1}}$. Continuing the same argument, it follows that $v^*_{r^*_l}\leq\tilde{v}_{r^*_l}$ for all $l\in\{k+1,\dots,\tilde{s}\}$, and hence the majorization inequality is violated for the last positive prize, which is impossible. ∎

Proof of Lemma 2

Using the derivatives $\tilde{U}_v=\phi'U_v$, $\tilde{U}_{vv}=\phi''U_v^2+\phi'U_{vv}$ and $\tilde{U}_{vve}=\phi'''U_v^2U_e+\phi''U_{vv}U_e+2\phi''U_vU_{ve}+\phi'U_{vve}$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial v}\frac{\tilde{U}_{ve}}{\tilde{U}_v}=\frac{1}{\tilde{U}_v^2}(\tilde{U}_{vve}\tilde{U}_v-\tilde{U}_{ve}\tilde{U}_{vv})=\frac{1}{\phi'^2U_v^2}\big[(\phi'''U_v^2U_e+\phi''U_{vv}U_e+2\phi''U_vU_{ve}+\phi'U_{vve})\phi'U_v\\&-(\phi''U_vU_e+\phi'U_{ve})(\phi''U_v^2+\phi'U_{vv})\big]=\frac{1}{U_v}\bigg[\frac{\phi'''}{\phi'}U_v^2U_e+\frac{\phi''}{\phi'}U_vU_{ve}+U_{vve}-\frac{\phi''^2}{\phi'^2}U_v^2U_e\\&-\frac{U_{ve}U_{vv}}{U_v}\bigg]=\frac{1}{U_v}\bigg[U_{vve}-\frac{U_{ve}U_{vv}}{U_v}+\frac{\phi''}{\phi'}U_vU_{ve}-U_v^2U_e\bigg(-\frac{\phi'''}{\phi'}+\frac{\phi''^2}{\phi'^2}\bigg)\bigg]\\&\leq\frac{1}{U_v}\bigg(U_{vve}-\frac{U_{ve}U_{vv}}{U_v}\bigg)=\frac{\partial}{\partial v}\frac{U_{ve}}{U_v},\end{aligned}$$

where the inequality follows because $U_{ve}\geq0$ and $\phi(\cdot)$ being NIARA is equivalent to $-\frac{\phi'''}{\phi'}+\frac{\phi''^2}{\phi'^2}\leq0$. ∎

Proof of Lemma 3

Let $A(\mathbf{v},e^*;\rho)$ denote the corresponding function A defined in (3). We know that $\mathbf{v}^1$ solves $\max_{\mathbf{v}\in\mathcal{V}_1}A(\mathbf{v},e^*;\rho_1)$ and satisfies $A(\mathbf{v}^1,e^*;\rho_1)=0$. By construction, $A(\mathbf{v},e^*;\rho_2)=$

$A(\frac{\rho_2}{\rho_1}\mathbf{v}, e^*; \rho_1)$, implying that $A(\mathbf{v}^2, e^*; \rho_2) = 0$. Moreover, $\max_{\mathbf{v}\in\mathcal{V}_b} A(\mathbf{v}, e^*; \rho_2)$ is solved by a $\mathbf{v}^*$ such that $\frac{\rho_2}{\rho_1}\mathbf{v}^* = \mathbf{v}^1$ when $\frac{\rho_2}{\rho_1}b = 1$. This produces the desired result. ■

Proof of Proposition 6

Similarly to the proof of Proposition 5, we first prove the following Lemma.

Lemma B3 *Suppose* $\tilde{e} \geq e^*$, $\beta' > \beta > 0$, $v' \in q(\beta', \lambda, e^*)$, $v \in q(\beta, \lambda, e^*)$, $\lambda > 0$, *and* $v' > v > 0$. *Then,*

$$\frac{n\beta' + \gamma(v', e^*)}{n\beta + \gamma(v, e^*)} \frac{U_v(v', e^*)}{U_v(v, e^*)} \geq \frac{n\beta' + \tilde{\gamma}(v', e^*)}{n\beta + \tilde{\gamma}(v, e^*)} \frac{\tilde{U}_v(v', e^*)}{\tilde{U}_v(v, e^*)} \geq \frac{n\beta' + \tilde{\gamma}(v', \tilde{e})}{n\beta + \tilde{\gamma}(v, \tilde{e})} \frac{\tilde{U}_v(v', \tilde{e})}{\tilde{U}_v(v, \tilde{e})}. \tag{39}$$

Proof The first inequality in (39) is the same as in Lemma B2 and is proven there. For the second inequality, note that

$$\begin{aligned}
&\frac{\partial}{\partial e} \frac{n\beta'\tilde{U}_v(v', e) + \tilde{U}_{ve}(v', e)}{n\beta\tilde{U}_v(v, e) + \tilde{U}_{ve}(v, e)} \propto \frac{n\beta'\tilde{U}_{ve}(v', e) + \tilde{U}_{vee}(v', e)}{n\beta'\tilde{U}_v(v', e) + \tilde{U}_{ve}(v', e)} - \frac{n\beta\tilde{U}_{ve}(v, e) + \tilde{U}_{vee}(v, e)}{n\beta\tilde{U}_v(v, e) + \tilde{U}_{ve}(v, e)} \\
&= \frac{n\beta' + \frac{\tilde{U}_{vee}(v', e)}{\tilde{U}_{ve}(v', e)}}{\frac{n\beta'}{\tilde{\gamma}(v', e)} + 1} - \frac{n\beta + \frac{\tilde{U}_{vee}(v, e)}{\tilde{U}_{ve}(v, e)}}{\frac{n\beta}{\tilde{\gamma}(v, e)} + 1} \leq 0,
\end{aligned}$$

since $\beta' > \beta > 0$, $v' > v$, $\tilde{\gamma}_v(v, e) \leq 0$ and $\frac{\partial}{\partial v} \frac{\tilde{U}_{vee}(v, e)}{\tilde{U}_{ve}(v, e)} \leq 0$. ■

The rest of the proof follows exactly the same steps as the proof of Proposition 5, except that Lemma B3 is used instead of Lemma B2. ■

Proof of Corollary 6

Apply Proposition 7 for $\tilde{u}(v) = u(bv)$, that is, check the effect of b on risk aversion:

$$\frac{\partial}{\partial b}\left(-\frac{u_{vv}(bv)}{u_v(bv)}\right) = \frac{\partial}{\partial b}\left(-b\frac{u''(bv)}{u'(bv)}\right) = \frac{\partial}{\partial c}\left(-c\frac{u''(c)}{u'(c)}\right),$$

where $c = bv$. ■

Proof of Lemma 4 It is well-known that the Tullock contest model can be derived using a tournament with multiplicative noise. Suppose output is $Y_i = e_i X_i$, where the support of X_i is an interval in $\mathbb{R}_{++}$. The additive noise model can be obtained by taking logs, $\log(Y_i) = \log(e_i) + \log(X_i)$. Additive noise $\log(X_i)$ is distributed with cdf $F^m(x) = F(\exp(x))$, and the utility function is transformed as $U^m(v, e_m) = U(v, \exp(e_m))$, where $e_m = \log(e)$ is the transformed effort. Letting β_r^m denote the coefficients β_r based on the transformed distribution $F^m(x)$, we obtain the following first-order condition from (3):

$$\sum_{r=1}^{n}\left[\beta_r^m U(v_r,e)+\frac{1}{n}U_e(v_r,e)e\right]=0. \tag{40}$$

Here, we used the transformation $U^m_{e_m}(v_r,e_m)=U_e(v,\exp(e_m))\exp(e_m)=U_e(v,e)e$.

For $U(v,e)=k(e)u(v)-c(e)$, the first-order condition (40) can be written as

$$\sum_{r=1}^{n}\left[\beta_r^m+\frac{k'(e)e}{nk(e)}\right]u(v_r)=\frac{c'(e)e}{k(e)}.$$

Comparing it to the first-order condition in the additive noise model with some utility function $\tilde{U}(v,e)=\tilde{k}(e)u(v)-\tilde{c}(e)$,

$$\sum_{r=1}^{n}\left[\beta_r^m+\frac{\tilde{k}'(e)}{n\tilde{k}(e)}\right]u(v_r)=\frac{\tilde{c}'(e)}{\tilde{k}(e)},$$

we can see that the two coincide if $\frac{k'(e)e}{k(e)}=\frac{\tilde{k}'(e)}{\tilde{k}(e)}$ and $\frac{c'(e)e}{k(e)}=\frac{\tilde{c}'(e)}{\tilde{k}(e)}$. It is straightforward to check that Eqs. (20) satisfy both conditions.

Tullock contests can be obtained using the multiplicative noise model with X_i following the Inverse Exponential distribution with parameter ξ. The corresponding additive noise, $\log(X_i)$, then follows the Gumbel distribution, which produces β_r^m as in the Lemma (Fu and Lu, 2012). ■