Maximal Matchings*

Anh Triêu† Iwan Bos‡ Marc Schröder§ Dries Vermeulen¶

February 13, 2025

Abstract

There are many situations where policymakers are primarily concerned with the availability and accessibility of goods or services. Examples include electricity, food, housing, medical supplies, *et cetera*. In such cases, the social goal may be to maximize the number of transactions, which we refer to as a maximal matching. This paper presents a mechanism that implements this objective. The mechanism satisfies the incentive and participation constraints, but requires external funding.

Keywords: Double Auction; Egalitarian Rule; Mechanism Design; Myerson's Lemma; Welfare Economics.

JEL Classification: C72; D47; D63.

*We appreciate the comments of and discussions with Christian Ewerhart, Floris Heukelom, Alexey Kushnir, Itzhak Rasooly, Christian Soegaard, participants at the 2024 Coalition Theory Network Workshop, the 2024 Oligo Workshop, the 7th world congress of the Game Theory Society (GAMES, 2024), and research seminar participants at Maastricht University and the Sapienza University of Rome. All errors and opinions are ours alone.

† Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: anh.trieu@maastrichtuniversity.nl

‡ Department of Organisation, Strategy, and Entrepreneurship, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: i.bos@maastrichtuniversity.nl

§ Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: m.schroder@maastrichtuniversity.nl

¶ Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: d.vermeulen@maastrichtuniversity.nl

"It is the greatest happiness of the greatest number that is the measure of right and wrong." (Bentham, 1776)

1 Introduction

Economics is often described as a social science that studies the production, distribution, and consumption of wealth.[1] Throughout its history, a good number of criteria have been developed to judge levels and distributions of welfare. Perhaps the most popular one is the so-called utilitarian rule. Roughly speaking, this rule says that goods and services should be produced and allocated in such a way that the combined utility of all agents involved is maximal.

The utilitarian rule has been criticized on many grounds, with the most common one being its neglect regarding issues of distributive justice.[2] Indeed, abiding by this principle may result in situations where most, if not all, wealth ends up in the hands of a few.[3] This can be particularly problematic when the goods or services are considered 'essential' such as electricity, food, housing, medical supplies, *et cetera*. In such cases, one can imagine policymakers to be more concerned with the availability and accessibility of wealth rather than with its aggregate value. That is to say, it may be preferred to have supplies produced and allocated in such a way that it serves as many parties as possible.

To fix ideas, consider the following example of two families looking for a house. Suppose there are two property owners, each of whom has an apartment available for sale. The apartments are virtually identical, but the costs of selling the place differ. After attending the open house, both families learn their valuation for the accommodation. Table 1 gives an overview of the valuations (v) and costs (c), which can be thought of as reservation prices. Family 1 values either apartment

v_1	v_2
10	5
c_1	c_2
2	6

Table 1: A market with 2 consumers and 2 producers.

[1] Definitions along these lines date back to *A Treatise on Political Economy; or the Production, Distribution, and Consumption of Wealth* by Jean-Baptiste Say (1803). For a detailed discussion, see Backhouse and Medema (2009).

[2] See, *e.g.*, Feldman (1987), Mandler (1999) and, more recently, Graafland (2022).

[3] As vividly formulated by Nozick (1974, p. 41): "*Utilitarian theory is embarrassed by the possibility of utility monsters who get enormously greater gains in utility from any sacrifice of others than these others lose. For, unacceptably, the theory seems to require that we all be sacrificed in the monster's maw, in order to increase total utility.*"

at 10, whereas family 2 values both at 5. The homeowners' costs are 2 and 6, respectively. Note that if the policymaker's goal is to maximize total surplus, it would like owner 1 to sell to family 1. This, however, implies that family 2 is left empty-handed, because owner 2's minimum price ($c_2 = 6$) exceeds this family's valuation ($v_2 = 5$). In this scenario, total surplus is $10 - 2 = 8$. Yet, if the policymaker's goal is to have housing for as many people as possible, then it would like owner 1 to trade with family 2 and owner 2 to sell to family 1. This yields a total surplus of $(10 - 6) + (5 - 2) = 7$ and a maximal number of matches, namely 2.

In this paper, we suppose that the social goal is to maximize the number of buyer-seller matches.[4] We coin such a matching-maximizing outcome a *maximal matching*. It is assumed that participation is voluntary, *i.e.*, neither consumers nor producers can be forced to engage in a transaction. What makes this problem challenging is that costs and valuations are private information. Following the above example, a family's actual valuation for a house is unknown to the owner as well as to competing buyers. Similarly, an owner often has a pretty good idea about the (opportunity) cost of selling his property, but that information is typically not available to any of the other market players. Moreover, both buyers and sellers are likely to pursue goals other than maximizing the total number of deals between them. This raises the question of whether one can design a mechanism that achieves a maximal matching. In what follows, we show that the answer to this question is affirmative.

We start our analysis by introducing an algorithm that enables the identification of a maximal matching. More specifically, as will become clear in the ensuing analysis, it selects the optimal matching-maximizing outcome in the sense that it yields the maximal matching with the highest total surplus. We then proceed by considering implementability. The key issue here is to specify a pricing rule that incentivizes all agents involved to reveal their actual valuations or costs. In other words, what price(s) do consumers have to pay and what payment(s) do producers need to receive to make truth-telling a dominant strategy? To answer this question, we show how one can apply the pricing rule as specified in Myerson's Lemma to both sides of the market.

In terms of desirable properties, the proposed mechanism yields an individual rational allocation with agents engaging in a transaction only when they find it in their interest to do so. That is, the mechanism satisfies the participation constraints. It also satisfies the incentive-

[4] It is worth emphasizing that the goal of matching maximization is not only applicable within the context of essential goods provision. In digital markets, for example, a platform owner may be primarily interested in maximizing the number of transactions or 'clicks' as this gives valuable information that can be exploited elsewhere.

compatibility constraints. Truth-telling thus constitutes an equilibrium and the objective of matching maximization is implementable. A notable drawback of the mechanism is that it is not budget balanced. A policymaker thus has to look for outside funds when it aims for a maximal matching.[5]

Our paper fits to the stream of literature on algorithmic mechanism design. The typical goal here is to design algorithms for mechanism design problems that satisfy a monotonicity constraint. It is precisely this monotonicity constraint that guarantees that agents are willing to reveal their information truthfully. The origins of algorithmic mechanism design date back to Nisan and Ronen (2001) and Lehmann *et al.* (2002).[6] However, not much is known yet about cases as described above, *i.e.*, situations where both sides of the market hold relevant private information. In this type of context, scholars have mainly considered other objectives such as revenue maximization (*e.g.*, Deshmukhl *et al.* (2002), Deng *et al.* (2014)) or budget-balancedness (*e.g.*, Colini-Baldeschi *et al.* (2016)).

Our analysis touches upon the classic efficiency-equity trade-off. While the utilitarian approach yields an efficient allocation of goods or services, matching maximization carries an egalitarian flavor. Indeed, as we show formally in Appendix A, the proposed mechanism with an equal-split of surpluses gives an outcome similar to what one would obtain by applying the egalitarian rule. This naturally relates to a growing body of recent work that shows how one may use market design to reach redistributive objectives when there is heterogeneity in marginal values for money.[7]

By maximizing the number of buyer-seller matches, the proposed mechanism achieves the social goal 'directly'. A policymaker, in principle, can achieve a comparable outcome also 'indirectly' through an appropriate (re-)distribution of agents' (initial) endowments. This, however, may prove problematic for at least two reasons. First, the information required to determine the preferred allocation is typically lacking and cannot be elicited truthfully. Second, redistribution methods (*e.g.*, taxes and subsidies) are in themselves inefficient. Apart from this, experimental research suggests free market competition to converge to a competitive equilibrium under plausible conditions, which commonly implies a less than maximal matching.[8] To achieve the latter

[5]A similar finding is obtained by Myerson and Satterthwaite (1983) in the context of bilateral trade. More generally, the mechanism design literature is rich in 'impossibility results' showing that it is typically impossible to design a mechanism that meets all desirable properties.

[6]For a survey of this literature, see Nisan (2015).

[7]See Dworczak, Kominers and Akbarpour (2021), Akbarpour, Dworczak and Kominers (2022) and Groh and Reuter (2023).

[8]See, *e.g.*, Smith (1962), Gode and Sunder (1993), Bosch-Domenech and Sunder (2000) and Lin *et al.* (2020).

then requires a central authority to provide the appropriate incentives. Finally, it is worth noting that indirect routes to maximal matchings may simply not be available due to technical limitations. For example, storing energy still proves challenging in many situations so that energy suppliers may have to serve their customers instantly and directly.

The next section introduces the modeling framework. Section 3 offers a formal definition of a maximal matching and presents an algorithm to identify such an allocation. We take up the issue of implementability in Section 4. Section 5 is devoted to the costs of matching maximization. Section 6 concludes. The link to egalitarianism is provided in Appendix A. All proofs are relegated to Appendix B.

2 The Model

Consider a market comprising a set of $M = \{1, \ldots, m\}$ consumers and a set of $N = \{1, \ldots, n\}$ producers. Each consumer $i \in M$ attempts to purchase one unit of a good, which it values at $v_i \geq 0$. Each producer $j \in N$ can produce one unit of this good at a cost of $c_j \geq 0$. There is a social planner which has the objective to meet society's aggregate preferences regarding welfare and its distribution. Since information is incomplete, the planner employs a direct mechanism to reach its goal.[9] To that end, it asks all consumers and producers to report their valuations and costs. Let $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ and $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ be the vectors of reported valuations and costs, respectively. Without loss of generality, it is assumed that $r_1 \geq r_2 \geq \ldots \geq r_m$ and $s_1 \leq s_2 \leq \ldots \leq s_n$. The vector $\mathbf{r}_{-i}$ indicates the reported valuations of all consumers other than i and the vector $\mathbf{s}_{-j}$ denotes the reported costs of all producers other than j. Let R and S be the corresponding sets of all possible reported valuation and cost profiles, respectively. Generic elements are represented by $\mathbf{r}, \mathbf{w} \in R$ and $\mathbf{s}, \mathbf{t} \in S$.

A direct mechanism, $\mathcal{M} = (A, p)$, consists of an *allocation rule* A and a *payment rule* p. An allocation rule $A : (\mathbf{r}, \mathbf{s}) \mapsto A(\mathbf{r}, \mathbf{s}) = (K, L)$ takes the profiles of communicated values as input and produces output $A(\mathbf{r}, \mathbf{s}) = (K, L)$, where $K \subseteq M$ and $L \subseteq N$ with $|K| \leq |L|$ indicate the buyers and sellers that are involved in a transaction. For each individual consumer and individual

[9] If information would be complete, then all valuations and costs are common priors. In that case, it is straightforward to select a mechanism that serves the social planner's goal. The adjective 'direct' refers to the revelation principle (see, *e.g.*, Gibbard (1973)), which states that for any mechanism equilibrium there exists an equivalent incentive-compatible direct revelation mechanism.

producer, the outcome of mechanism A is then, respectively, given by:

$$a_i(\mathbf{r}, \mathbf{s}) = \begin{cases} 1 \text{ if } i \in K \\ 0 \text{ if } i \notin K, \end{cases}$$

and

$$a_j(\mathbf{r}, \mathbf{s}) = \begin{cases} 1 \text{ if } j \in L \\ 0 \text{ if } j \notin L. \end{cases}$$

For $A = (K, L)$, total utility is defined as the difference between the valuations of all matched consumers and the costs of all matched producers. Formally,

$$TU(\mathbf{v}, \mathbf{c}, \mathbf{r}, \mathbf{s}) = \sum_{i \in K} v_i - \sum_{j \in L} c_j, \text{ where } A(\mathbf{r}, \mathbf{s}) = (K, L).$$

The *payment rule* also takes all reported values and costs as an input and uses this information to specify the amount $p_i(\mathbf{r}, \mathbf{s})$ that consumer i has to pay and the amount $p_j(\mathbf{r}, \mathbf{s})$ that producer j receives.

In what follows, let $T \subseteq M \times N$ be a *matching*, where for every $i \in M$, $|\{j|(i,j) \in T\}| \leq 1$, and for every $j \in N$, $|\{i|(i,j) \in T\}| \leq 1$. Note that an allocation rule only indicates which market players are involved in a trade, whereas a matching additionally specifies which consumers and producers are linked together.

We consider voluntary matchings only, *i.e.*, neither consumers nor producers can be forced to take part in a transaction. To make this concrete, we now introduce three closely related concepts.

Definition 2.1 *A match $(i,j) \in T$ is **value-creating** if $v_i \geq c_j$.*

Definition 2.2 *A matching T is **bilaterally rational** if all $(i,j) \in T$ are value-creating. Let $\mathcal{T}_{BR}$ be the set of all bilaterally rational matchings.*

Definition 2.3 *A mechanism (A, p) is **individually rational** if for all $i \in M$, $j \in N$, for all $\mathbf{r} = (r_1, \ldots, r_m) \in R$ and $\mathbf{s} = (s_1, \ldots, s_n) \in S$, $r_i \cdot a_i(\mathbf{r}, \mathbf{s}) \geq p_i(\mathbf{r}, \mathbf{s})$ and $s_j \cdot a_j(\mathbf{r}, \mathbf{s}) \leq p_j(\mathbf{r}, \mathbf{s})$.*

Note that bilateral rationality is a necessary condition for a matching to be voluntary. To see this, suppose there is a matching that is not bilaterally rational. In that case, there is a pair (i,j) for which it holds that $v_i < c_j$. If $p_i > v_i$, then consumer i does not want to buy. When $p_i \leq v_i < c_j$, however, producer j does not want to sell since the payment is insufficient to cover its cost. Absent bilateral rationality, there is thus no price that allows for voluntary participation by both buyer i and seller j simultaneously. Finally, individual rationality implies that an agent's gain of participating weakly exceeds its gain of not participating for all realizations. An individually rational mechanism thus induces voluntary participation.

3 Matching Maximization

We now employ the above model to examine matching maximization. We start by offering a precise definition of a matching-maximizing outcome, which we refer to as a *maximal matching*. We then present an algorithm to identify such an outcome. We conclude this section by showing that the proposed algorithm is optimal in the sense that it selects the utility-maximizing matching among all maximal matchings.

Definition 3.1 *A matching $T \in \mathcal{T}_{BR}$ is **maximal** when*

$$|T| = \max_{Q \in \mathcal{T}_{BR}} |Q|.$$

The set of all maximal matchings is given by $\mathcal{T}_M$.

The next example illustrates these concepts.

Example 3.2 *Consider a market with 4 consumers and 4 producers, all of which report their actual valuationks and costs. Table 2 presents their reports, in an ordered fashion.*

r_1	r_2	r_3	r_4
9	8	6	3
s_1	s_2	s_3	s_4
4	5	7	11

Table 2: A market with 4 consumers and 4 producers.

Observe that consumer 4 and producer 4 cannot engage in a value-creating transaction and that this market has a maximum of 3 value-creating pairs. Therefore, $|T| = 3$ for any maximal matching T and $A = (\{1,2,3\},\{1,2,3\})$. Notice, however, that A need not yield 3 value-creating matches. For example, $T_1 = \{(1,3),(2,2),(3,1)\}$ or $T_2 = \{(1,1),(2,3),(3,2)\}$ does, but $T_3 = \{(1,1),(2,2),(3,3)\}$ does not. As a result, $T_1, T_2 \in \mathcal{T}_M$ and $T_3 \notin \mathcal{T}_M$.

Flip Algorithm

Knowing what maximal matchings look like, a natural next question is how to find them. To that end, we now introduce the *Flip Algorithm*, which is an algorithm that identifies a matching with a maximal number of value-creating pairs.

Recall that the consumers' reported valuations are in decreasing order, *i.e.*, $r_1 \geq r_2 \geq \ldots \geq r_m$, and that the producers' reported costs are in increasing order, *i.e.*, $s_1 \leq s_2 \leq \ldots \leq s_n$. The Flip Algorithm then computes the following matching. Let $\bar{k}$ be the largest index k such that

$r_i \geq s_{k+1-i}$ for all $1 \leq i \leq k$. Match $\bar{k}$ pairs $\{(i, \bar{k}+1-i) \mid 1 \leq i \leq \bar{k}\}$. We denote the resulting matching by T_F.

The next example gives an illustration.

Example 3.3 *We apply the Flip Algorithm to the market as described in Example 3.2 above. Starting with $k = 1$, one compares the valuation of the first consumer with the cost of the first producer. Since $r_1 \geq s_1$, we proceed with $k = 2$. We then reverse the order of the first two producers in Table 2 and obtain Table 3 below. Since $r_1 \geq s_2$ and $r_2 \geq s_1$, we proceed with $k = 3$. Reversing the order of the first three producers in Table 2 gives the situation as presented in Table 4. Also in this case, the first three pairs are value-creating so that we proceed with $k = 4$. This results in the situation as presented in Table 5. Note that with $k = 4$ valuations fall short of costs for the first and the forth pair. The Flip Algorithm, therefore, yields $A = (\{1,2,3\},\{1,2,3\})$ and $T_F = \{(1,3),(2,2),(3,1)\}$.*

r_1	r_2	r_3	r_4
9	8	6	3
s_2	s_1	s_3	s_4
5	4	7	11

Table 3: Valuations and costs when $k = 2$.

r_1	r_2	r_3	r_4
9	8	6	3
s_3	s_2	s_1	s_4
7	5	4	11

Table 4: Valuations and costs when $k = 3$.

r_1	r_2	r_3	r_4
9	8	6	3
s_4	s_3	s_2	s_1
11	7	5	4

Table 5: Valuations and costs when $k = 4$.

The following result establishes a general property of a matching, which is useful in showing that the Flip Algorithm yields a matching-maximizing outcome.

Lemma 3.4 *Let T be a matching with k matches. If there is a pair $(i,j) \in T$ with $i + j < k+1$, then there is a pair $(i',j') \in T$ with $i' > i$ and $i + j' \geq k+1$.*

To see that the matching resulting from the Flip Algorithm is indeed maximal, suppose there is another maximal matching T consisting of $\bar{k}$ matches. If consumer i is matched with some producer $j > \bar{k}+1-i$ under T, then it can be matched with seller $j = \bar{k}+1-i$ since $r_i \geq s_j \geq s_{\bar{k}+1-i}$. If consumer i is matched with some producer $j < \bar{k}+1-i$ under T, then there is a consumer $i' > i$ that is matched with a producer $j' \geq \bar{k}+1-i$ (Lemma 3.4). Therefore,

$r_i \geq r_{i'} \geq s_{j'} \geq s_{\bar{k}+1-i}$ so that consumer i can be matched with seller $j = \bar{k}+1-i$. This implies that it is possible to construct the situation resulting from the Flip Algorithm without reducing the number of matches. Since T is a maximal matching, so is the matching induced by the Flip Algorithm. The next theorem summarizes this finding.

Theorem 3.5 *The Flip Algorithm yields a matching in $\mathcal{T}_M$.*

Optimal Maximal Matching

The above theorem shows that the Flip Algorithm identifies a matching-maximizing outcome. This algorithm does not just select any such matching, however. Indeed, it identifies the optimal maximal matching in the sense that it selects the matching-maximizing outcome with the highest total utility.

Theorem 3.6 *The Flip Algorithm maximizes total utility over $\mathcal{T}_M$.*

4 Implementability

The preceding section explains how one can identify an optimal matching-maximizing outcome. A key question is then whether there exists an incentive-compatible mechanism to attain such a matching. Is it implementable, that is? In this section, we show that the answer is in the affirmative.

4.1 Preliminaries

Let us first introduce some concepts that we use to establish implementability. As a starter, we explain what it means for a mechanism to be *dominant strategy incentive compatible* (DSIC) in our setting.

Definition 4.1 *A mechanism is **DSIC** if, for all $i \in N$, $r_i, w_i \in \mathbb{R}_+$ and $\mathbf{r}_{-i} \in \mathbb{R}_+^{m-1}$, it holds that:*

$$r_i \cdot a_i(r_i, \mathbf{r}_{-i}, \mathbf{s}) - p_i(r_i, \mathbf{r}_{-i}, \mathbf{s}) \geq r_i \cdot a_i(w_i, \mathbf{r}_{-i}, \mathbf{s}) - p_i(w_i, \mathbf{r}_{-i}, \mathbf{s}).$$

And, for all $j \in M$, $s_j, t_j \in \mathbb{R}_+$ and $\mathbf{s}_{-j} \in \mathbb{R}_+^{n-1}$, it holds that:

$$p_j(s_j, \mathbf{s}_{-j}, \mathbf{r}) - s_j \cdot a_j(s_j, \mathbf{s}_{-j}, \mathbf{r}) \geq p_j(t_j, \mathbf{s}_{-j}, \mathbf{r}) - s_j \cdot a_j(t_j, \mathbf{s}_{-j}, \mathbf{r}).$$

If a mechanism is DSIC, then the net gain of reporting the actual valuation or cost weakly exceeds the net gain of reporting anything else. This makes reporting truthfully a dominant strategy for both buyers and sellers. Using this concept, implementability is then defined as follows:

Definition 4.2 *An allocation rule A is **implementable** if there exists a payment rule p such that (A, p) is DSIC.*

Given the reported valuations of all consumers other than i, we write $a_i(r_i, \mathbf{r}_{-i}, \mathbf{s})$ as $a_i(r_i)$ and $a_i(w_i, \mathbf{r}_{-i}, \mathbf{s})$ as $a_i(w_i)$, for all $r_i, w_i \in \mathbb{R}_+$. In the same manner, we write $p_i(r_i, \mathbf{r}_{-i}, \mathbf{s})$ and $p_i(w_i, \mathbf{r}_{-i}, \mathbf{s})$ as $p_i(r_i)$ and $p_i(w_i)$, respectively for all $r_i, w_i \in \mathbb{R}_+$. We then have the following basic observation.

Lemma 4.3 *Fix $\mathbf{r}_{-i}$. If a mechanism $(A,\ p)$ is DSIC, then*

$$w_i \cdot a_i(w_i) - w_i \cdot a_i(r_i) \geq p_i(w_i) - p_i(r_i) \geq r_i \cdot a_i(w_i) - r_i \cdot a_i(r_i).$$

This result reveals that when a mechanism is DSIC, the associated allocation rule is monotonic.

Definition 4.4 *An allocation rule A is **monotonic** if for all i, $r_i, w_i \in \mathbb{R}_+$ and $\mathbf{r}_{-i} \in \mathbb{R}_+^{m-1}$, it holds that:*

$$w_i \cdot a_i(w_i) - w_i \cdot a_i(r_i) \geq r_i \cdot a_i(w_i) - r_i \cdot a_i(r_i).$$

If an allocation rule is monotonic, then a consumer does not receive less when it would raise its reported valuation. In a similar vein, a producer does not sell less when it would reduce its reported cost.

We now have all the ingredients available to (re-)state Myerson's Lemma.

Lemma 4.5 (Myerson's Lemma (1981)) *For a single-parameter environment, the following three properties hold:*

1. *An allocation rule A is implementable if, and only if, A is monotonic.*
2. *If A is monotonic, then there is a unique payment rule such that the mechanism (A, p) is DSIC.*
3. *The payment rule in 2 is given by an explicit formula. For each consumer i:*

   $$p_i(r_i, \mathbf{r}_{-i}, \mathbf{s}) = \int_0^{r_i} z dA(z).$$

   Intuitively, consumers pay their 'switch point' where they go from 'not being matched' to 'being matched'.

4.2 Findings

In the following, we apply Myerson's Lemma to both sides of the market. To facilitate the analysis, let A_F be the allocation that one obtains by applying the Flip Algorithm and let p_F be the corresponding payment rule. We refer to (A_F, p_F) as the *Flip Mechanism.* Using the above introduced concepts and definitions, we will now: (i) prove that the allocation rule associated with the Flip Algorithm is implementable, (ii) determine the actual payments, and (iii) show that the Flip Mechanism is individually rational.

To begin, we prove that the allocation rule A_F is monotonic. Once monotonicity is established, Myerson's Lemma provides the pricing scheme that implements the allocation rule.

Proposition 4.6 *The allocation rule A_F is monotonic and, therefore, implementable.*

Remark 4.7 *There exist so-called 'greedy algorithms' capable of identifying a maximal matching. Yet, such algorithms are typically not monotonic and, therefore, not implementable. To illustrate, consider the following, relatively simple, algorithm:*

1. *Order all consumers' reported valuations in a decreasing order.*
2. *Order all producers' reported costs in a decreasing order.*
3. *Starting with consumer 1, match consumer i to the first producer j for which it holds that $r_i \geq s_j$. Repeat this process, while bypassing any buyer that cannot be matched.*

Consider a market with 2 consumers and 2 producers. Table 6 presents their reported valuations and costs in an ordered fashion. The greedy algorithm matches consumer 1 to producer 1 and

r_1	r_2
10	5
s_1	s_2
8	6

Table 6: A market with 2 consumers and 2 producers.

then stops. Suppose, however, that producer 2 reports a cost of $s_2' = 9$ instead. In that case, it would be matched, which violates monotonicity.

Knowing that A_F is implementable, we now apply Myerson's lemma to determine the actual payments. Proposition 4.8 specifies the prices.

Proposition 4.8 *The payment rule p_F is given by:*

1. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r}_{-i},\mathbf{s})| = \bar{k}$*, then* $p_i = r_{\bar{k}+1}$*.*

2. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r}_{-i},\mathbf{s})| + 1$ *and* $|T_F(\mathbf{r}_{-i},\mathbf{s})| = |T_F(\mathbf{r}_{-i},\mathbf{s}_{-1})|$*, then* $p_i = s_1$*.*

3. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r}_{-i},\mathbf{s})| + 1 = \bar{k}$ *and* $|T_F(\mathbf{r}_{-i},\mathbf{s})| = |T_F(\mathbf{r}_{-i},\mathbf{s}_{-1})| + 1$*, then* $p_i = r_{\bar{k}}$*.*

4. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r},\mathbf{s}_{-j})| = \bar{k}$*, then* $p_j = s_{\bar{k}+1}$*.*

5. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r},\mathbf{s}_{-j})| + 1$ *and* $|T_F(\mathbf{r},\mathbf{s}_{-j})| = |T_F(\mathbf{r}_{-1},\mathbf{s}_{-j})|$*, then* $p_j = r_1$*.*

6. *If* $|T_F(\mathbf{r},\mathbf{s})| = |T_F(\mathbf{r},\mathbf{s}_{-j})| + 1 = \bar{k}$ *and* $|T_F(\mathbf{r},\mathbf{s}_{-j})| = |T_F(\mathbf{r}_{-1},\mathbf{s}_{-j})| + 1$*, then* $p_j = s_{\bar{k}}$*.*

The next result is implied by the preceding analysis.

Corollary 4.9 *The Flip Mechanism* (A_F, p_F) *is individually rational.*

Let us conclude this section with an example that illustrates the payment rule.

Example 4.10 *Imagine a market with 6 consumers and 6 producers. The communicated valuations and costs are presented in Table 7 below. Notice that* $|T_F| = 5$ *with and without consumer 1. When this consumer would indeed leave the market, it is replaced by consumer 6 which reports a valuation of 1.5. Consequently, consumer 1 goes from 'not being matched' to 'being matched' when reporting a valuation of 1.5, which constitutes its 'switch point'. In this case, therefore, consumer 1 pays 1.5.*

r_1	r_2	r_3	r_4	r_5	r_6
9	8	7	5	4	1.5
s_1	s_2	s_3	s_4	s_5	s_6
1	2	3	6	7	10

Table 7: A market where consumer 1 pays $r_{\bar{k}+1} = 1.5$.

Now consider the same market with one notable difference; consumer 6 communicates a valuation of 0.5 instead of 1.5. Table 8 contains the reported valuations and costs. Notice that $|T_F| = 5$*, but that* $|T_F| = 4$ *without consumer 1. Moreover,* $|T_F| = 4$ *without consumer 1 and producer 1. Since producer 1 has the lowest reported cost among all vacant producers, consumer 1 goes from 'not being matched' to 'being matched' when reporting a valuation of* s_1*. In this case, therefore, consumer 1 pays 1.*

Finally, consider the same market as depicted in Table 7, but again with one difference; producer 5 now reports a cost of 8.5 instead of 7. Table 9 contains the communicated valuations and costs.

r_1	r_2	r_3	r_4	r_5	r_6
9	8	7	5	4	0.5
s_1	s_2	s_3	s_4	s_5	s_6
1	2	3	6	7	10

Table 8: A market where consumer 1 pays $s_1 = 1$.

As before, $|T_F| = 5$, whereas $|T_F| = 4$ when consumer 1 would leave the market. Moreover, $|T_F| = 3$ without consumer 1 and producer 1. Absent consumer 1, the result of the Flip Algorithm would be such that consumer 5 is matched with producer 1. Hence, consumer 1 goes from 'not being matched' to 'being matched' when reporting a valuation of r_5. In this case, therefore, consumer 1 pays 4.

r_1	r_2	r_3	r_4	r_5	r_6
9	8	7	5	4	0.5
s_1	s_2	s_3	s_4	s_5	s_6
1	2	3	6	8.5	10

Table 9: Market where consumer 1 pays $r_{\bar{k}} = 4$.

5 The Price of Matching Maximization

Thus far, we have introduced an algorithm to identify the optimal maximal matching and shown implementability of the associated allocation rule. One may then wonder what it costs to actually implement it. It is this question that we now turn to.

5.1 Total Utility Maximization: A Benchmark

To evaluate the cost of matching maximization, we use total utility maximization as a benchmark. The *TU-maximizing Algorithm* identifies a matching such that the total utility of all agents is maximal. It works as follows.

Recall that the reports of all consumers are in decreasing order, *i.e.*, $r_1 \geq r_2 \geq \ldots r_m$, and that the reports of all producers are in increasing order, *i.e.*, $s_1 \leq s_2 \leq \ldots s_n$. Let k be an index such that $r_l > s_l$ implies $l \leq k$ and $r_l < s_l$ implies $l > k$. Match k pairs $\{(i,i) \mid 1 \leq i \leq k\}$.

To see that this rule indeed maximizes total utility, note that $r_l \leq s_l$ for any $l > k$ so that matching k pairs generates (weakly) more utility than matching more than k pairs. Moreover, $r_l \geq s_l$ for any $l \leq k$ so that matching up to k pairs (weakly) increases the total value-created. Note further that the TU-maximizing Algorithm yields a matching that is bilaterally rational.

Indeed, since $r_1 \geq r_2 \geq \ldots \geq r_m$ and $s_1 \leq s_2 \leq \ldots \leq s_n$ and the first k pairs are matched, it holds that $r_i \geq r_k \geq s_k \geq s_j$, for all $i \leq k$ and $j \leq k$.

Example 5.1 *To illustrate, consider the market as described in Example 3.2 above. The reported valuations and costs are presented in Table 10, which is the same as Table 2. Note that the 'break-*

r_1	r_2	r_3	r_4
9	8	6	3
s_1	s_2	s_3	s_4
4	5	7	11

Table 10: A market with 4 consumers and 4 producers.

even point' is at $k = 2$ *since the reported cost* $s_3 = 7$ *exceeds the reported valuation* $r_3 = 6$ *for the third pair. In this example, therefore,* $A = (\{1,2\},\{1,2\})$ *and the TU-maximizing Algorithm yields* 2 *pairs.*

It can be easily verified that a TU-maximizing allocation rule is monotonic and, therefore, implementable. Hence, it is possible to apply the payment rule as given by Myerson's Lemma. In fact, as is well-known, one can use a VCG-mechanism to achieve an allocatively efficient outcome.[10] Yet, this type of mechanism is typically not budget-balanced.[11] The next two results confirm that a VCG-mechanism indeed (almost) always creates a deficit, also in our setting.

Proposition 5.2 *If the policymaker adopts a VCG-mechanism, then* $\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} p_j$.

Proposition 5.3 *The VCG-mechanism yields a deficit if, and only if, the allocation rule is unique.*

Taken together, this means that total utility maximization more often than not requires external funding (*e.g.*, a subsidy).[12]

5.2 How (In)efficient is Matching Maximization?

Using the TU maximization results as reference, we now address the question of what it costs to maximize the number of value-creating matches. We consider two measures of efficiency: (i) the amount of deficit, and (ii) the number of matches. Let us discuss each in turn.

[10] VCG-mechanisms are named after the contributions of Vickrey (1961), Clarke (1971) and Groves (1973).

[11] See, *e.g.*, Vickrey (1961) and Myerson and Satterthwaite (1983).

[12] As shown by McAfee (1992), one way to achieve budget balance is by sacrificing one value-creating match.

5.2.1 Amount of Deficit

How (in)efficient is matching maximization in monetary terms? The following theorem sheds some first light on this question.

Theorem 5.4 *Consider two implementable allocation rules A and A' where $K \supseteq K'$ and $L \supseteq L'$, for all $\mathbf{r} \in R$ and $\mathbf{s} \in S$. Denote the corresponding payment rules by p and p', respectively. Then, $p_i \leq p'_i$ for all consumers i and $p_j \geq p'_j$ for all producers j.*

The implication of this result is that the amount of deficit is (weakly) larger in case of matching maximization. This is so because an implementable matching-maximizing allocation rule yields at least the same number of matches as the TU-maximizing allocation rule. Therefore, each consumer pays less, whereas each producer gets more. The following corollary summarizes this finding.

Corollary 5.5 *The deficit under an implementable matching-maximizing mechanism is at least as large as under a VCG-mechanism.*

It is worth emphasizing that this may hold even when the number of matches is the same under both mechanisms. This is illustrated by the next example.

Example 5.6 *Consider a market with 4 consumers and 4 producers. Table 11 contains the reported valuations and costs. Note that both the VCG Mechanism and the Flip Mechanism yield*

r_1	r_2	r_3	r_4
9	8	7	1
s_1	s_2	s_3	s_4
2	3	4	10

Table 11: A market with 4 consumers and 4 producers.

3 matches. Under VCG, the deficit is 9 since each matched consumer pays 4 and each matched producer obtains 7. Under the Flip Mechanism, the deficit is 21 since each matched consumer pays 2 and each matched producer obtains 9. Hence, the deficit under the Flip Mechanism is larger than under the VCG Mechanism even though both give rise to 3 matches.

Knowing that matching maximization creates a (larger) deficit, one may wonder about its magnitude. How big can it be? The next proposition provides an upper bound.

Proposition 5.7 *Suppose that $|T_F| = k$. The deficit is at most $k \cdot (r_1 - s_1)$.*

5.2.2 Number of Matches

An alternative way to assess the (in)efficiency of matching maximization is by analyzing the difference in the amount of transactions. As the example in the introduction of this paper illustrates, matching maximization may yield twice the number of transactions in comparison to TU maximization. As the next proposition shows, however, this is also the maximum ratio that can be attained.[13]

Proposition 5.8 *The ratio of the number of matched pairs under matching maximization to the number of matched pairs under TU maximization is at most 2.*

It is worth noting that this finding extends to an incomplete bipartite graph. To see this, suppose that potential trading partners face some constraints. For example, a buyer and seller located far away from each other should perhaps not be matched since transportation costs would be too high. Imagine, then, an incomplete bipartite graph that indicates all feasible matches. To identify a maximal matching, one can proceed as follows. Assign flow 1 to each edge between consumer i's vertex and producer j's vertex when $r_i - s_j \geq 0$ and remove the edge when $r_i - s_j < 0$. A maximal matching on the resulting graph can then be found by applying Ford-Fulkerson's algorithm.

To identify a matching that maximizes total utility, one can proceed in a similar fashion. Assign flow $f_{ij} = r_i - s_j$ to each edge between consumer i's vertex and producer j's vertex when $r_i - s_j \geq 0$ and remove the edge when $r_i - s_j < 0$. One can then select the weighted maximum matching for the resulting graph. Now consider the resulting graph under matching maximization. To add one more match that maximizes the weight, one has to eliminate at most two matched pairs. Matching maximization will therefore not yield more than twice the number of matches under TU maximization.

[13] This result also follows from known results in graph theory that the size of a maximal matching is at least half the size of a maximum matching.

6 Concluding Remarks

In the same year that Adam Smith launched his *Wealth of Nations* (1776), the prominent British scholar Jeremy Bentham published *A Fragment on Government* (1776). It is in this essay that Bentham formulated what became the fundamental maxim of utilitarianism, namely: "it is the greatest happiness of the greatest number that is the measure of right and wrong". Of course, when taken literally, the constructs 'greatest happiness' and 'greatest number' need not be reconcilable and pursuing one may well come at the cost of the other. While welfare economics has been predominantly concerned with the first by exploring traits and conditions that maximize the 'size of the pie', we focused on the second by asking how to maximize 'the number of pie bakers'.

Under the assumption that no market player can be forced to trade, we introduced an algorithm that identifies a matching with the maximum number of transactions. More specifically, we showed that this algorithm selects the optimal maximal matching in the sense that it creates the greatest happiness *conditional on* the greatest number. Importantly, the objective of matching maximization is implementable and we presented a mechanism that implements it. Doing so literally comes at a price, however, because the mechanism does not satisfy the property of budget balance. A policymaker thus needs to find external funding if it seeks to maximize the number of matches.

Appendix A: A Justification for Maximal Matchings

With its focus on maximizing the number of traded goods or services, matching maximization has an egalitarian flavor. The goal of this appendix is to make this more precise. In what follows, we consider and compare matchings' surplus distributions. It is shown that one obtains the most egalitarian outcome by applying the Flip Algorithm and have each matched pair divide its value-created equally.

To facilitate the analysis, we start with the following definition.

Definition A.1: *Consider a matching T. For each match $(i, j) \in T$, let $r_i - s_j = a_i + b_j$, where a_i and b_j are the surplus shares allocated to consumer i and producer j, respectively. Moreover, $a_i = 0$ for any unmatched consumer and $b_j = 0$ for any unmatched producer.*

Clearly, there are potentially many ways in which the value-created can be allocated. In what follows, let $\mathbf{d} = (a_1, \ldots, a_m, b_1, \ldots, b_n)$ be the surplus distribution vector. The *ordered* surplus distribution vector of $\mathbf{d}$ is $\hat{\mathbf{d}} = (d_1, \ldots, d_{m+n})$, where $d_i \leq d_j$ for all $1 \leq i \leq j \leq m + n$. Hence, the elements of $\hat{\mathbf{d}}$ are organized in a nondecreasing order.

To compare ordered surplus distributions, we use the following definition.

Definition A.2: *Consider two vectors $\mathbf{d}, \mathbf{e} \in \mathbb{R}^{m+n}$. The vector $\mathbf{d}$ is **lexicographically equal** to $\mathbf{e}$ when $\mathbf{d} = \mathbf{e}$. The vector $\mathbf{d}$ is **lexicographically larger** than $\mathbf{e}$ when $d_i > e_i$ for $i = \min\{j \in \{1, \ldots, m + n\} \mid d_j \neq e_j\}$. That is, $\mathbf{d}$ is lexicographically larger than $\mathbf{e}$ when the first coordinate that differs is higher under $\mathbf{d}$.*

Let us now establish the link between matching maximization and the egalitarian rule. Specifically, we show that the ordered surplus distribution that one obtains by maximizing the minimum utility is the same as the ordered surplus distribution that one obtains by applying the Flip Algorithm and allocating the resulting surpluses equally.

Proposition A.3 *Suppose that $r_i > s_j$ for all $(i, j) \in T_F$. The ordered surplus distribution that results from the Flip Algorithm $T_F(\mathbf{r}, \mathbf{s})$, with $a_i = b_j = \frac{r_i - s_j}{2}$ for any $(i, j) \in T_F$, is lexicographically larger or equal than the ordered surplus distribution of any other matching $T(\mathbf{r}, \mathbf{s})$.*

Appendix B: Proofs

Proof of Lemma 3.4

Let T be a matching with k matches. Suppose that there is a match $(i,j) \in T$ with $i+j < k+1$ and no match $(i',j') \in T$ with $i' > i$ and $i+j' \geq k+1$. We show that $|T| \leq k = 1$. Since there is no match $(i',j') \in T$ with $i' > i$ and $i+j' \geq k+1$, any consumer $i' \geq i$ is matched to a producer j' where $j' < k-i+1$. So, $j' \leq k-i$. Hence, there are at most $k-i$ matches. Moreover, there are at most $i-1$ matches from the first $i-1$ consumers. Taken together, this implies at most $k-i+i-1 = k-1$ matches. ∎

Proof of Theorem 3.6

Given $(\mathbf{r},\mathbf{s})$, consider some maximal matching $T \in \mathcal{T}_M$ other than T_F, *i.e.*, $T \neq T_F$. Then, there is a consumer i such that $(i,j) \notin T_F$, for all j, but there is a producer j such that $(i,j) \in T$. As the Flip Algorithm matches the first $\bar{k}$ pairs, this implies $i > \bar{k}$. Moreover, since $|T_F| = |T|$, there is a consumer i' such that $(i',j) \notin T$, for all j, but there is a producer j' such that $(i',j') \in T_F$. As the Flip Algorithm matches the first $\bar{k}$ pairs, this implies $i' \leq \bar{k}$. Since $i' \leq \bar{k} < i$, it follows that $r_{i'} \geq r_i$. A similar argument applies when there is a producer j that is matched under T, but not under T_F. We conclude that the total utility with T_F is weakly larger than with T. ∎

Proof of Lemma 4.3

If the true valuation is r_i and the mechanism is DSIC, then it holds that:

$$r_i \cdot a_i(r_i) - p_i(r_i) \geq r_i \cdot a_i(w_i) - p_i(w_i) \iff p_i(w_i) - p_i(r_i) \geq r_i \cdot a_i(w_i) - r_i \cdot a_i(r_i). \quad (1)$$

Likewise, if the true valuation is w_i, then:

$$p_i(w_i) - p_i(r_i) \leq w_i \cdot a_i(w_i) - w_i \cdot a_i(r_i). \quad (2)$$

Combining (1) and (2), one obtains Lemma 4.3. ∎

Proof of Proposition 4.6

Recall that $|T_F| = \bar{k}$ and suppose that $a_i(r_i) = 1$. Fixing all other valuations and costs, we show that if consumer i reports $w_i \geq r_i$ instead, then $a_i(w_i) = 1$.

Since $a_i(r_i) = 1$, consumer i belongs to the first $\bar{k}$ matched consumers. If consumer i would report $w_i \geq r_i$ instead, then there are at least $\bar{k}$ matched pairs. To see this, consider a maximal matching $T \in \mathcal{T}_M$ in which consumer i is matched with producer j. Since all other valuations and costs remain the same, there still are $\bar{k}-1$ value-creating pairs when consumer i raises its reported valuation to $w_i \geq r_i$. As this consumer is matched when reporting r_i, it holds that $r_i \geq s_j$. Consequently, this consumer can be matched with the same producer when reporting w_i instead, because $w_i \geq r_i \geq s_j$. Therefore, the number of matches does not decrease when a consumer increases its reported valuation, all else equal.

As consumer i belongs to the first $\bar{k}$ matched pairs when reporting r_i, it also belongs to the first $\bar{k}$ matched pairs when reporting $w_i \geq r_i$. Therefore, $a_i(w_i) = 1$. A similar logic applies to the supply side. We conclude that the allocation rule A_F is monotonic. Implementability then directly follows from Myerson's Lemma. ■

Proof of Proposition 4.8

Suppose that T_F consists of $\bar{k}$ matches. In the following, we prove cases 1, 2 and 3. A similar logic applies to the supply side, *i.e.*, cases 4, 5 and 6.

1. A first possibility is that the Flip Algorithm would still yield $\bar{k}$ transactions when some matched consumer i leaves the market, *i.e.*, $|T_F(\mathbf{r}, \mathbf{s})| = |T_F(\mathbf{r}_{-i}, \mathbf{s})| = \bar{k}$. Suppose that some matched consumer i would indeed leave. In that case, it is replaced by consumer $\bar{k}+1$, because the Flip Algorithm yields $\bar{k}$ matches both with and without consumer i. Following Myerson's Lemma, a consumer has to pay its 'switch point' where it goes from 'not being matched' to 'being matched'. Hence, consumer i has to report a valuation of $r_{\bar{k}+1}$ to replace consumer $\bar{k}+1$. In this case, therefore, $p_i = r_{\bar{k}+1}$.

2. A second possibility is that the Flip Algorithm yields $\bar{k}-1$ transactions when some matched consumer i leaves the market, *i.e.*, $|T_F(\mathbf{r}, \mathbf{s})| = |T_F(\mathbf{r}_{-i}, \mathbf{s})| + 1$, and that there are still $\bar{k}-1$ transactions when producer 1 would not be present either, *i.e.*, $|T_F(\mathbf{r}_{-i}, \mathbf{s})| = |T_F(\mathbf{r}_{-i}, \mathbf{s}_{-1})|$. If so, then producer 1 reports the lowest cost among all vacant producers. Hence, consumer i has to report a valuation of s_1 to go from 'not being matched' to 'being matched'. In this case, therefore, $p_i = s_1$.

3. A third possibility is that the Flip Algorithm yields $\bar{k}-1$ transactions when some matched consumer i leaves the market, *i.e.*, $|T_F(\mathbf{r}, \mathbf{s})| = |T_F(\mathbf{r}_{-i}, \mathbf{s})| + 1$, and that there are $\bar{k}-2$ transactions when producer 1 would not be present either, *i.e.*, $|T_F(\mathbf{r}_{-i}, \mathbf{s})| = |T_F(\mathbf{r}_{-i}, \mathbf{s}_{-1})| + 1$. If producer 1 is in the market, then it is matched to the $\bar{k}_{th}$ consumer. Hence, consumer i has to report a valuation of $r_{\bar{k}}$ to go from 'not being matched' to 'being matched'. In this case, therefore, $p_i = r_{\bar{k}}$. ■

Proof of Corollary 4.9

By Myerson's Lemma, a consumer that is involved in a transaction has to pay its 'switch point' when it goes from 'not being matched' to 'being matched' and zero otherwise. Consider some consumer i with reported valuation r_i and switch point w_i. If $a_i(r_i) = 0$, then $r_i \cdot a_i(r_i) = 0 = p_i(a_i(r_i))$. If $a_i(r_i) = 1$, then $r_i \geq w_i$ as consumer i is matched. Thus, $r_i \cdot a_i(r_i) = r_i \geq w_i = p_i(a_i(w_i))$. Taken together, therefore, it holds that $r_i \cdot a_i(r_i) \geq p_i(a_i(r_i))$. A similar argument applies to the supply side. We conclude that the Flip Mechanism is individually rational. ■

Proof of Proposition 5.2

Following the generic logic of VCG-mechanisms, consumers pay their externality. To determine the price for some matched consumer i, suppose that the TU-maximizing Algorithm yields k matches and is applied again without this consumer. If $r_{k+1} \geq s_k$, then the producer that was matched with consumer i before will be matched again. In this case, there are still k matches. If $r_{k+1} < s_k$, then the exclusion of consumer i implies the exclusion of some producer j. In this case, there are $k-1$ matches. Taken together, this means that any matched consumer pays a price $p_i = \max\{r_{k+1}, s_k\}$. By the same token, each matched producer obtains a price $p_j = \min\{s_{k+1}, r_k\}$.

Since the TU-maximizing Algorithm yields k matches, it holds that $s_k \leq r_k$ and $r_{k+1} < s_{k+1}$. Hence, $\max\{r_{k+1}, s_k\} \leq \min\{s_{k+1}, r_k\}$ and therefore:

$$\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} p_j = k \cdot \max\{r_{k+1}, s_k\} - k \cdot \min\{s_{k+1}, r_k\} \leq 0.$$

Consequently, if the policymaker adopts a VCG-mechanism, then $\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} p_j$. ■

Proof of Proposition 5.3

Suppose that the VCG-mechanism yields k matches. Let us first show that the allocation rule is unique when there is a deficit. To that end, assume that the allocation rule is not unique. We derive a contradiction. If the allocation rule is not unique, then there are at least two agents with rank k or $k+1$ that report the same value. Following the proof of Proposition 5.2, any matched consumer i pays $p_i = \max\{r_{k+1}, s_k\}$ and any matched producer j obtains $p_j = \min\{r_k, s_{k+1}\}$. Using the fact that $r_{k+1} \leq r_k$, $r_{k+1} < s_{k+1}$, $s_k \leq s_{k+1}$, and $s_k \leq r_k$, this effectively leaves two possibilities.

1. If $s_k \leq r_k = r_{k+1} < s_{k+1}$, then $p_i = \max\{r_{k+1}, s_k\} = r_{k+1}$ and $p_j = \min\{r_k, s_{k+1}\} = r_k$. Hence, $p_i = p_j$ for all $i \in K$ and $j \in L$. In this case, therefore, there is no deficit.

2. If $r_{k+1} < s_k = s_{k+1} \leq r_k$, then $p_i = \max\{r_{k+1}, s_k\} = s_k$ and $p_j = \min\{r_k, s_{k+1}\} = s_{k+1}$. Hence, $p_i = p_j$ for all $i \in K$ and $j \in L$. In this case, therefore, there is no deficit.

We conclude that if there is a deficit, then the allocation rule is unique.

Let us now show that there is a deficit when there is a unique allocation rule. Suppose, by contradiction, that there is a unique allocation rule, but no deficit. By the preceding analysis, all matched consumers pay $\max\{r_{k+1}, s_k\}$ and all matched producers obtain $\min\{r_k, s_{k+1}\}$. If $\max\{r_{k+1}, s_k\} = r_{k+1}$, then there is a deficit since $r_k > r_{k+1}$ and $s_{k+1} > r_{k+1}$. If $\max\{r_{k+1}, s_k\} = s_k$, then there is a deficit since $s_k < r_k$ by uniqueness and $s_k < s_{k+1}$. We conclude that if the allocation rule is unique and $r_k > s_k$, then there is a deficit. ■

Proof of Theorem 5.4

As A and A' are implementable, they are monotonic (Myerson's Lemma). Consider some consumer i with 'switch point' r_i under $A(\mathbf{r}, \mathbf{s})$ and 'switch point' w_i under $A'(\mathbf{r}, \mathbf{s})$. Let us now view the allocation of consumer i with valuation w_i under $A(\mathbf{r}, \mathbf{s})$. Since $K \supseteq K'$ for all $\mathbf{r} \in R$, $\mathbf{s} \in S$, it holds that $a_i(w_i) \geq a'_i(w_i)$ and, therefore, $w_i \geq r_i$. Since A is implementable by payment rule p, each consumer pays its 'switch point' where it goes from 'not being matched' to 'being matched'. Similarly for A'. Therefore, $p'_i = w_i \geq r_i = p_i$.

Now consider some producer j with 'switch point' s_j under $A(\mathbf{r}, \mathbf{s})$ and 'switch point' t_j under $A'(\mathbf{r}, \mathbf{s})$. Let us now view the allocation of producer j with cost t_j under $A(\mathbf{r}, \mathbf{s})$. Since $L \supseteq L'$ for all $\mathbf{r} \in R$, $\mathbf{s} \in S$, it holds that $a_j(t_j) \geq a'_j(t_j)$ and, therefore, $t_j \leq s_j$. Since A is implementable by payment rule p, each producer obtains its 'switch point' where it goes from 'not being matched' to 'being matched'. Similarly for A'. Therefore, $p'_j = t_j \leq s_j = p_j$. ■

Proof of Proposition 5.7

The lowest possible price for a matched consumer is s_1, whereas the highest possible price for a matched producer is r_1. Since there are k matches, the greatest possible deficit is $k \cdot (r_1 - s_1)$. ■

Proof of Proposition 5.8

To begin, note that the example in the introduction of this paper shows that the ratio can be 2. Let us now show that it cannot be more than 2. To that end, suppose that the TU-maximizing Algorithm yields k matches, which implies $r_{k+m} < s_{k+m}$ for all $m \geq 1$. Suppose further that the Flip Algorithm yields $2k + m$ matches, where $m \geq 1$. We derive a contradiction. If the Flip Algorithm leads to $2k + m$ matches, then consumer 1 is matched with producer $2k + m$ and consumer 2 is matched with producer $2k + m - 1$. Following this logic, consumer $k + 1$ is matched with producer $k+m$, which implies $r_{k+1} \geq s_{k+m} \geq s_{k+1}$. This, however, contradicts the fact that $r_{k+1} < s_{k+1}$. ■

Proof of Proposition A.3

To begin, note that the ordered surplus distribution with any bilaterally rational matching is lexicographically larger than the ordered surplus distribution of a matching that is not bilaterally rational. In case of the latter, there is at least one agent that obtains a 'negative surplus', whereas all agents receive a weakly positive surplus when the matching is bilaterally rational. Hence, since the Flip Algorithm yields a bilaterally rational matching, the resulting ordered surplus distribution is lexicographically larger than the ordered surplus distribution of any matching that is not bilaterally rational.

Next, consider a bilateral matching T, where $|T| < |T_F|$. Since each match has a strictly positive surplus and unmatched agents receive zero surplus, it follows immediately that the ordered surplus distribution that results from the Flip Algorithm is lexicographically larger than the ordered surplus distribution of any matching with strictly fewer matches.

Finally, note that the ordered surplus distribution that results from the Flip Algorithm with an equal division of surplus is lexicographically larger than the ordered surplus distribution that results from the Flip Algorithm with an unequal division of surplus. It remains to be shown that it is also lexicographically larger or equal than the ordered surplus distribution of any other maximal matching. To that end, consider another maximal matching $T \in \mathcal{T}_M$. Since both T and T_F are maximal matchings the number of unmatched agents is the same. Consider a pair $(i^*, j^*) \in T_F$ that is least positive value-creating. We claim that there exists a pair $(i', j') \in T$ such that either $r_{i'} = r_{i^*}$ and $s_{j'} = s_{j^*}$, or $r_{i'} - s_{j'} < r_{i^*} - s_{j^*}$. If $(i^*, j^*) \in T$, then the claim is true by setting $i' = i^*$ and $j' = j^*$. If $(i^*, j) \in T$ and $j > j^*$, we know that $s_j \leq s_{j^*}$. Hence, $r_{i^*} - s_j \leq r_{i^*} - s_{j^*}$. Then, take $i' = i^*$ and $j' = j$. If $(i^*, j) \in T$ and $j < j^*$, then there is a pair (i'', j'') such that $i'' > i^*$ and $i^* + j'' \geq k+1$ (Lemma 3.4). Since $r_{i^*} \geq r_{i''}$ and $s_{j''} \geq s_{j^*}$, it holds that $r_{i''} - s_{j''} \leq r_{i^*} - s_{j^*}$. Hence, take $i' = i''$ and $j' = j''$.

If the inequality in the claim applies, we know that the Flip Algorithm with an equal division of surplus lexicographically maximizes the vector of ordered surplus distributions. Otherwise, we consider $T_{F,1} = T_F \setminus \{(i^*, j^*)\}$ and $T_1 = T \setminus \{(i', j')\}$. We continue the procedure until there exist a pair $(i^*, j^*) \in T_{F,n}$ and $(i', j') \in T_n$ where $r_{i'} - s_{j'} < r_{i^*} - s_{j^*}$ for some $n \geq 1$, or until $T_{F,\bar{k}} = \varnothing$. Hence, the Flip Algorithm with an equal division of surplus lexicographically maximizes the vector of ordered surplus distributions. ■

References

Backhouse, Roger E. and Steven G. Medema (2009), "Retrospectives: On the Definition of Economics," *Journal of Economic Perspectives*, 23(1), 221-233;

Clarke, Edward H. (1971), "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33;

Colini-Baldeschi, Ricardo, Bart de Keijzer, Stefano Leonardi and Stefano Turchetta (2016), "Approximately Efficient Double Auctions with Strong Budget Balance," *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, 1424-1443;

Deng, Xiaotie, Paul Goldberg, Bo Tang and Jinshan Zhang (2014), "Revenue Maximization in a Bayesian Double Auction Market," *Theoretical Computer Science*, 539, 1-12;

Deshmukh, Kaustubh, Andrew V. Goldberg, Jason D. Hartline and Anna R. Karlin (2002), "Truthful and Competitive Double Auctions," *Algorithms—ESA 2002*, 10, 361-373;

Dworczak, Piotr, Scott Duke Kominers and Mohammad Akbarpour (2021), "Redistribution through Markets," *Econometrica*, 89(4), 1665–1698;

Feldman, Allan M. (1987), "Welfare Economics," published in: *The New Palgrave: A Dictionary of Economics*, John Eatwell, Murray Milgate and Peter Newman (eds.), first edition;

Gibbard, Allan (1973), "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41(4), 587–601;

Gode, Dhananjay K. and Shyam Sunder (1993), "Allocative Efficiency of Markets with Zero-Intelligence Traders: Market as a Partial Substitute for Individual Rationality," *Journal of Political Economy*, 101(1), 119-137;

Graafland, Johan (2022), "*Ethics and Economics: An Introduction to Free Markets, Equality and Happiness*," Routledge: New York;

Groh, Carl-Christian and Marco Reuter (2023), "Mechanism Design for Unequal Societies," *ZEW Discussion Paper*, 23-050;

Groves, Theodore (1973), "Incentives in Teams," *Econometrica*, 41(4), 617–631;

Lehmann, Daniel, Liadan I. O'Callaghan and Yoav Shoham (2002), "Truth Revelation in Approximately Efficient Combinatorial Auctions," *Journal of the ACM*, 49(5), 577–602;

Lin, Po-Hsuan, Alexander L. Brown, Taisuke Imai, Joseph Tao-yi Wang, Stephanie W. Wang and Colin F. Camerer (2020), "Evidence of General Economic Principles of Bargaining and Trade from 2000 Classroom Experiments," *Nature Human Behaviour*, 4, 917-927;

Mandler, Michael (1999), "*Dilemmas in Economic Theory: Persisting Foundational Problems of Microeconomics*," Oxford University Press: New York;

McAfee, R. Preston (1992), "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56(2), 434–450;

Myerson, Roger B. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6(1), 58–73;

Myerson, Roger B. and Mark A. Satterthwaite (1983), "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29(2), 265–281;

Nisan, Noam (2015), "Algorithmic Mechanism Design: Through the Lens of Multiunit Auctions," *Handbook of Game Theory with Economic Applications*, 4, 477–515;

Nisan, Noam and Amir Ronen (2001), "Algorithmic Mechanism Design," *Games and Economic Behavior*, 35, 166–196;

Nozick, Robert (1974), "*Anarchy, State, and Utopia*," Blackwell Publishers: Oxford;

Smith, Vernon L. (1962), "An Experimental Study of Competitive Market Behavior," *Journal of Political Economy*, 70(2), 111-137;

Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16(1), 8–37.